

# A First Order Free Lunch for SQRT-Lasso\*

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## Abstract

Many statistical machine learning techniques sacrifice convenient computational structures to gain estimation robustness and modeling flexibility. In this paper, we study this fundamental tradeoff through a SQRT-Lasso problem for sparse linear regression and sparse precision matrix estimation in high dimensions. We explain how novel optimization techniques help address these computational challenges. Namely, we propose a pathwise iterative smoothing shrinkage thresholding algorithm for solving the SQRT-Lasso optimization problem, and provide a novel model-based perspective for analyzing the smoothing optimization framework, which allows us to establish a near linear convergence (R-linear convergence) guarantee for our proposed algorithm, without sacrificing statistical accuracy. This implies that solving the SQRT-Lasso optimization problem is almost as easy as solving the Lasso optimization problem, while the former requires much less parameter tuning effort. Moreover, we show that our proposed algorithm can also be applied to sparse precision matrix estimation, and enjoys desirable computational as well as statistical properties. Numerical experiments are provided to support our theory.

## 1 Introduction

Given a design matrix  $X \in \mathbb{R}^{n \times d}$  and a response vector  $y \in \mathbb{R}^n$ , we consider a linear model  $y = X\theta^* + \epsilon$ , where  $\theta^* \in \mathbb{R}^d$  is an unknown coefficient vector, and  $\epsilon \in \mathbb{R}^n$  is a random noise vector with i.i.d. sub-Gaussian entries, where  $\mathbb{E}[\epsilon_i] = 0$  and  $\mathbb{E}[\epsilon_i^2] = \sigma^2$  for all  $i = 1, \dots, n$ . We are interested in estimating  $\theta^*$  in high dimensions where  $n/d \rightarrow 0$ . A popular assumption in high dimensions is that only a small subset of variables are relevant in modeling, i.e., many entries of  $\theta^*$  are zero. To get such a sparse estimator, Tibshirani (1996) propose Lasso, which solves

$$\bar{\theta} = \underset{\theta}{\operatorname{argmin}} \frac{1}{n} \|y - X\theta\|_2^2 + \lambda_{\text{Lasso}} \|\theta\|_1, \quad (1.1)$$

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where  $\lambda_{\text{Lasso}} > 0$  is a regularization parameter that encourages the solution sparsity. The statistical properties of Lasso have been established in [Zhang and Huang \(2008\)](#); [Zhang \(2009\)](#); [Bickel et al. \(2009\)](#); [Wainwright \(2009\)](#); [Meinshausen and Yu \(2009\)](#); [Negahban et al. \(2012\)](#). In particular, given  $\lambda_{\text{Lasso}} \asymp \sigma \sqrt{\log d/n}$ , the Lasso estimator in (1.1) attains the minimax optimal rate of convergence in parameter estimation<sup>1</sup>,

$$\|\bar{\theta} - \theta^*\|_2 = \mathcal{O}_P \left( \sigma \sqrt{\frac{s^* \log d}{n}} \right), \quad (1.2)$$

where  $s^*$  denotes the number of nonzero entries in  $\theta^*$  ([Ye and Zhang, 2010](#); [Raskutti et al., 2011](#)).

Despite these favorable properties, the Lasso approach has a significant drawback: The selected regularization parameter  $\lambda_{\text{Lasso}}$  scales linearly with the *unknown* quantity  $\sigma$ . Therefore, we often need to carefully tune  $\lambda_{\text{Lasso}}$  over a wide range of potential values in order to get a good finite-sample performance. To overcome this drawback, [Belloni et al. \(2011\)](#) propose SQRT-Lasso, which solves

$$\bar{\theta} = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \frac{1}{\sqrt{n}} \|y - X\theta\|_2 + \lambda_{\text{SQRT}} \|\theta\|_1. \quad (1.3)$$

They further show that SQRT-Lasso requires no prior knowledge of  $\sigma$ . Choosing  $\lambda_{\text{SQRT}} \asymp \sqrt{\log d/n}$ , the SQRT-Lasso estimator in (1.3) attains the same optimal statistical rate of convergence as (1.2). This means that the regularization selection for SQRT-Lasso does not scale with the unknown  $\sigma$ . We can easily specify a smaller range of potential values for tuning  $\lambda_{\text{SQRT}}$  than Lasso.

In addition to estimating  $\theta^*$ , SQRT-Lasso can also be used to estimate  $\sigma$ , which further makes it applicable to sparse precision matrix estimation; this is not the case with Lasso. Specifically, given  $n$  i.i.d. observations sampled from a  $d$ -variate normal distribution with mean 0 and a sparse precision matrix  $\Theta^*$ , [Liu and Wang \(2012\)](#) propose an estimator based on SQRT-Lasso (See more details in §4), and show that it attains the minimax optimal statistical rate of convergence in parameter estimation,

$$\|\bar{\Theta} - \Theta^*\|_2 = \mathcal{O}_P \left( \|\Theta^*\|_2 \cdot s^* \sqrt{\frac{\log d}{n}} \right),$$

where  $\|\Theta^*\|_2$  denotes the spectral norm of  $\Theta^*$  (i.e., the largest singular value of  $\Theta^*$ ), and  $s^*$  denotes the maximum number of nonzero entries in each column of  $\Theta^*$  (i.e.  $\max_j \sum_i \mathbb{1}(\Theta_{ij}^* \neq 0) \leq s^*$ ).

Though SQRT-Lasso simplifies the tuning efforts and achieves the optimal statistical properties for both sparse linear regression and sparse precision matrix estimation in high dimensions, the optimization problem in (1.3) is computationally more challenging than (1.1) for Lasso. This is because the  $\ell_2$  loss in SQRT-Lasso does not have the same nice computational structures as the least square loss in Lasso. Namely, the  $\ell_2$  loss can be nondifferentiable, and does not have a Lipschitz continuous gradient. [Belloni et al. \(2011\)](#) convert (1.3) to a second order cone optimization

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<sup>1</sup>The notation  $\mathcal{O}_P(\cdot)$  is defined formally at the end of this section

problem, and further solve it by an interior point method; Li et al. (2015) solve (1.3) by an alternating direction method of multipliers (ADMM) algorithm. Neither of these algorithms, however, can scale to large problems. In contrast, Xiao and Zhang (2013) propose an efficient pathwise iterative shrinkage thresholding algorithm (PISTA) to solve (1.1) for Lasso, which attains a linear convergence to the unique sparse global optimum with high probability.

To address this computational challenge of SQRT-Lasso, we propose a pathwise iterative smoothing shrinkage thresholding algorithm (PIS<sup>2</sup>TA) to solve (1.3). Specifically, we apply the conjugate dual smoothing approach to the nonsmooth  $\ell_2$  loss (Nesterov, 2005; Beck and Teboulle, 2012), and obtain a smooth surrogate denoted by  $\|y - X\theta\|_\mu$ , where  $\mu > 0$  is a smoothing parameter (See more details in §2). We then apply PISTA to solve the partially smoothed optimization problem as follows:

$$\bar{\theta} = \operatorname{argmin}_{\theta \in \mathbb{R}^d} \frac{1}{\sqrt{n}} \|y - X\theta\|_\mu + \lambda_{\text{SQRT}} \|\theta\|_1. \quad (1.4)$$

Existing computational theory guarantees that our proposed algorithm attains a sublinear convergence to the global optimum in term of the objective value (Nesterov, 2005). This is because the existing computational analyses of the conjugate dual smoothing approach do not take certain specific modeling structures into consideration. For example: (I) The  $\ell_2$  loss is only nonsmooth when all residuals are equal to zero (significantly overfitted). But this is very unlikely to happen because we are solving (1.3) with a sufficiently large regularization; (II) Although the smoothed  $\ell_2$  loss is not strongly convex, if we restrict the solution to a sparse domain, the smoothed  $\ell_2$  loss can behave like a strongly convex function over a neighborhood of  $\theta^*$ . Moreover, our numerical experiments show that PIS<sup>2</sup>TA achieves far better empirical computational performance than sublinear convergence for solving SQRT-Lasso, and is nearly as efficient as PISTA for solving Lasso. This is significantly more efficient than other competing algorithms (Belloni et al., 2011; Li et al., 2015).

Motivated by these observations, we establish a new computational theory for PIS<sup>2</sup>TA, which exploits the above modeling structures. Particularly, we show that PIS<sup>2</sup>TA achieves a near linear convergence (R-linear convergence<sup>2</sup>) to the unique sparse global optimum for solving (1.3) with high probability, and also gives us a well fitted model. There are two implications: (I) We can solve the SQRT-Lasso optimization as nearly efficiently as solving the Lasso optimization; (II) We pay almost no price in optimization accuracy when using the smoothing approach for solving the SQRT-Lasso optimization, because (1.4) and (1.3) share the same unique sparse global optimum with high probability, over the data model generation distribution.

As an extension of our theory for the SQRT-Lasso optimization, we further analyze the computational properties of PIS<sup>2</sup>TA for sparse precision matrix estimation in high dimensions. We show that PIS<sup>2</sup>TA also achieves an R-linear convergence to the unique sparse global optimum with

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<sup>2</sup>Here, R-linear convergence means that the algorithm attains a linear convergence rate after a finite (polynomial) number of iterations.

high probability. We provide numerical experiments on both synthetic and real data to support our theory. All proofs of our analysis are deferred to the supplementary material.

**Notations:** Given a vector  $v = (v_1, \dots, v_d)^\top \in \mathbb{R}^d$ , we define vector norms:  $\|v\|_1 = \sum_j |v_j|$ ,  $\|v\|_2^2 = \sum_j v_j^2$ , and  $\|v\|_\infty = \max_j |v_j|$ . We denote the number of nonzero entries in  $v$  as  $\|v\|_0 = \sum_j \mathbb{1}(v_j \neq 0)$  and the subvector of  $v$  with the  $j$ -th entry removed as  $v_{\setminus j} = (v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_d)^\top \in \mathbb{R}^{d-1}$ . Let  $\mathcal{I} \subseteq \{1, \dots, d\}$  be an index set and  $\bar{\mathcal{I}}$  be the complementary set to  $\mathcal{I}$ , i.e.  $\bar{\mathcal{I}} = \{j \mid j \in \{1, \dots, d\}, j \notin \mathcal{I}\}$ . We use  $v_{\mathcal{I}}$  to denote a subvector of  $v$  by extracting all entries of  $v$  with indices in  $\mathcal{I}$ . Given a matrix  $A \in \mathbb{R}^{d \times d}$ , we use  $A_{*j} = (A_{1j}, \dots, A_{dj})^\top$  to denote the  $j$ -th column and  $A_{k*} = (A_{k1}, \dots, A_{kd})^\top$  to denote the  $k$ -th row. Let  $\Lambda_{\max}(A)$  and  $\Lambda_{\min}(A)$  be the largest and smallest eigenvalues of  $A$  respectively. We define  $\|A\|_F^2 = \sum_j \|A_{*j}\|_2^2$  and  $\|A\|_2 = \sqrt{\Lambda_{\max}(A^\top A)}$ . We denote  $A_{\setminus i \setminus j}$  as the submatrix of  $A$  with the  $i$ -th row and the  $j$ -th column removed. We denote  $A_{\setminus ij}$  ( $A_{i \setminus j}$ ) as the  $j$ -th column ( $i$ -th row) of  $A$  with its  $i$ -th entry ( $j$ -th entry) removed. Given an index set  $\mathcal{I} \subseteq \{1, \dots, d\}$ , we use  $A_{\mathcal{I}\mathcal{I}}$  to denote a submatrix of  $A$  by extracting all entries of  $A$  with both row and column indices in  $\mathcal{I}$ . We denote  $A > 0$  if  $A$  is a positive-definite matrix. Given two real sequences  $\{A_n\}, \{a_n\}$ ,  $A_n = \mathcal{O}(a_n)$  (or  $A_n = \Omega(a_n)$ ) if and only if  $\exists M \in \mathbb{R}^+$  and  $N \in \mathbb{N}$  such that  $|A_n| \leq M|a_n|$  (or  $|A_n| \geq M|a_n|$ ) for all  $n \geq N$ .  $A_n \asymp a_n$  if  $A_n = \mathcal{O}(a_n)$  and  $A_n = \Omega(a_n)$  simultaneously.  $A_n = \mathcal{O}_p(a_n)$  if  $\forall \delta \in (0, 1)$ ,  $\exists M \in \mathbb{R}^+$  and  $N_\delta \in \mathbb{N}$  such that  $\mathbb{P}[|A_n| > M|a_n|] < \delta$  for all  $n \geq N_\delta$ .  $A_n = o(a_n)$  if  $\forall \delta > 0$ ,  $\exists N_\delta \in \mathbb{N}$  such that  $|A_n| \leq \delta|a_n|$  for all  $n \geq N_\delta$ , i.e.,  $\lim_{n \rightarrow \infty} A_n/a_n = 0$ .

## 2 Algorithm

Our proposed algorithm consists of three components: (I) Conjugate Dual Smoothing, (II) Iterative Shrinkage Thresholding Algorithm (ISTA), and (III) Pathwise Optimization.

(I) **Conjugate Dual Smoothing** is adopted to obtain a smooth surrogate of  $\ell_2$  loss (Nesterov, 2005; Beck and Teboulle, 2012). We denote the smoothed  $\ell_2$  loss function as

$$\|y - X\theta\|_\mu = \max_{\|z\|_2 \leq 1} z^\top (y - X\theta) - \frac{\mu}{2} \|z\|_2^2. \quad (2.1)$$

The optimization problem in (2.1) admits a closed form solution:

$$\mathcal{L}_\mu(\theta) = \frac{1}{\sqrt{n}} \|y - X\theta\|_\mu = \begin{cases} \frac{1}{2\mu\sqrt{n}} \|y - X\theta\|_2^2, & \text{if } \|y - X\theta\|_2 < \mu \\ \frac{1}{\sqrt{n}} \|y - X\theta\|_2 - \frac{\mu}{2}, & \text{o.w.} \end{cases}.$$

We present several two-dimensional examples of the smoothed  $\ell_2$  norm using different  $\mu$ 's in Figure 1. A larger  $\mu$  introduces a larger approximation error, but makes the approximation smoother. We then consider the following partially smoothed optimization problem,

$$\bar{\theta} = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \mathcal{F}_{\mu, \lambda}(\theta), \text{ where } \mathcal{F}_{\mu, \lambda}(\theta) = \mathcal{L}_\mu(\theta) + \lambda \|\theta\|_1. \quad (2.2)$$

(II) **Iterative Shrinkage Thresholding Algorithm** (ISTA) is applied to solve (2.2) (Nesterov, 2013). Particularly, given  $\theta^{(t)}$ , the estimator at the  $t$ -th iteration of ISTA, we consider the quadratic

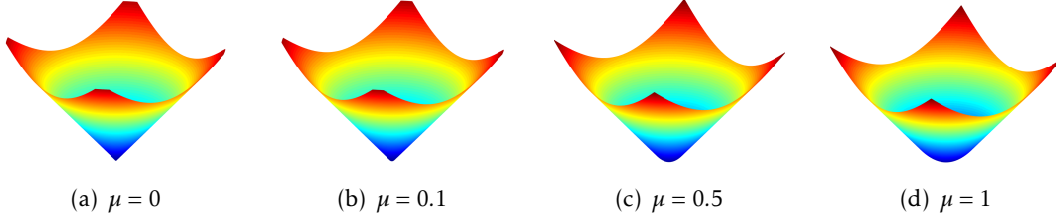


Figure 1: Examples of  $\|x\|_2$  ( $\mu = 0$ ) and  $\|x\|_\mu$  with  $\mu = 0.1, 0.5$ , and  $1$  respectively for  $x \in \mathbb{R}^2$ .

approximation of  $\mathcal{F}_{\mu,\lambda}(\theta)$  at  $\theta = \theta^{(t)}$ ,

$$\mathcal{Q}_{\mu,\lambda}(\theta, \theta^{(t)}) = \mathcal{L}_\mu(\theta^{(t)}) + \nabla \mathcal{L}_\mu(\theta^{(t)})^\top (\theta - \theta^{(t)}) + \frac{L^{(t+1)}}{2} \|\theta - \theta^{(t)}\|_2^2 + \lambda \|\theta\|_1, \quad (2.3)$$

where  $L^{(t+1)}$  is a step size parameter determined by backtracking line search. We then take

$$\theta^{(t+1)} = \underset{\theta}{\operatorname{argmin}} \mathcal{Q}_{\mu,\lambda}(\theta, \theta^{(t)}) = \mathcal{G}_{\lambda/L^{(t+1)}}(\theta^{(t)} - \nabla \mathcal{L}_\mu(\theta^{(t)})/L^{(t+1)}),$$

where for  $\tau > 0$ ,  $\mathcal{G}_\tau(x) = [\operatorname{sign}(x_j) \max\{|x_j| - \tau, 0\}]_{j=1}^d$  is the soft thresholding operator. For simplicity, we denote

$$\theta^{(t+1)} = \mathcal{T}_{L^{(t+1)}, \mu, \lambda}(\theta^{(t)}).$$

Given a prespecified precision  $\varepsilon$ , we terminate the iterations of ISTA when the approximate KKT condition holds:

$$\omega_{\mu,\lambda}(\theta^{(t)}) = \min_{g \in \partial \|\theta^{(t)}\|_1} \|\nabla \mathcal{L}_\mu(\theta^{(t)}) + \lambda g\|_\infty \leq \varepsilon. \quad (2.4)$$

(III) **Pathwise Optimization** is essentially a multistage optimization scheme for boosting the computational performance. We solve (2.2) using a geometrically decreasing sequence of regularization parameters  $\lambda_{[1]} > \dots > \lambda_{[N]}$ , where  $\lambda_{[N]} = \lambda_{\text{SQRT}}$ , and a geometrically decreasing sequence of smoothing parameters  $\mu_1 > \dots > \mu_{[N]}$ , where  $\mu_{[N]}$  is sufficiently small. This yields a sequence of output solutions  $\widehat{\theta}_{[1]}, \dots, \widehat{\theta}_{[N]}$  from sparse to dense, where we use the subscript  $[K]$  to denote the  $K$ -th optimization stage and  $\widehat{\theta}_{[K]}$  satisfies  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K]}) \leq \varepsilon_{[K]}$  for prespecified  $\varepsilon_{[K]}$  for all  $K = 1, \dots, N$ .

Particularly, at the  $K$ -th optimization stage, we choose  $\widehat{\theta}_{[K-1]}$  (the output solution of the  $(K-1)$ -th stage) as the initial solution, i.e.,  $\theta_{[K]}^{(0)} = \widehat{\theta}_{[K-1]}$ , and solve (2.2) using ISTA, with  $\lambda = \lambda_{[K]}$ ,  $\mu = \mu_{[K]}$ , and the termination criteria as  $\varepsilon_{[K]}$  in (2.4). This is also referred as warm start initialization in existing literature (Friedman et al., 2007; Zhao et al., 2014).

We summarize our proposed algorithm, PIS<sup>2</sup>TA, in Algorithm 1, and a flowchart for the structure of PIS<sup>2</sup>TA is provided in Figure 2. The overall algorithm contains three structurally nested loops:

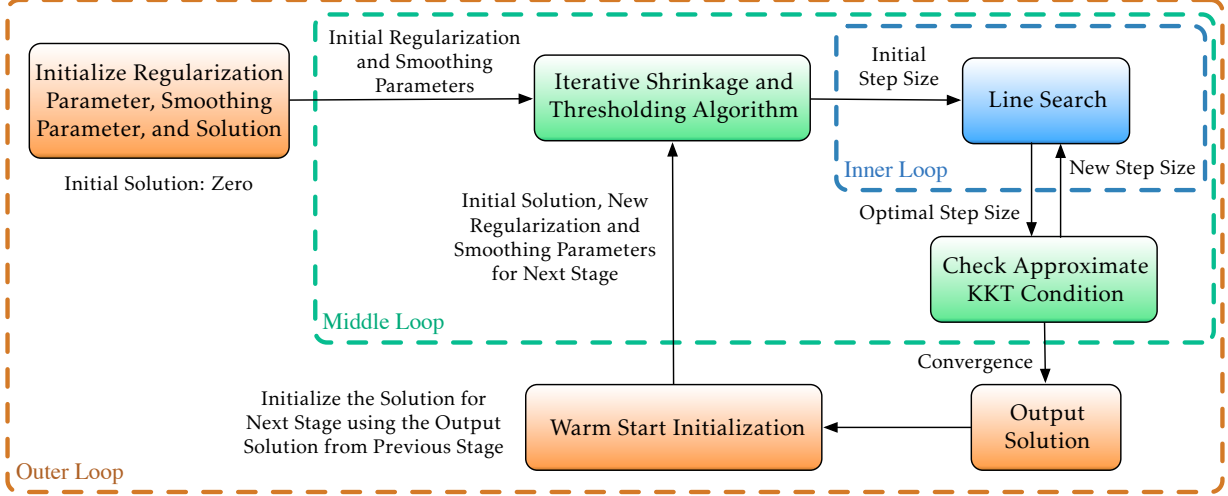


Figure 2: PIS<sup>2</sup>TA for SQRT-Lasso with 3 nested loops: (1) warm start initialization; (2) iterative shrinkage and thresholding algorithm; (3) backtracking line search.

- (1) Outer loop: The warm start initialization, also known as the pathwise optimization scheme, is adopted to estimate  $\theta$  from sparse to dense using a decreasing sequence of regularization parameters  $\{\lambda_{[K]}\}_{K=1}^N$  and a decreasing sequence of smoothing parameters  $\{\mu_{[K]}\}_{K=1}^N$ .
- (2) Middle loop: The iterative shrinkage thresholding algorithm (ISTA) is adopted to solve (2.2) based on the smoothed  $\ell_2$  norm and quadratic approximation of the objective function, with the optimal step size parameter  $L_{[K]}^{(t)}$  obtained via the backtracking line search.
- (3) Inner loop: The backtracking line search is adopted to find the optimal step size parameter  $L_{[K]}^{(t)}$  at the  $t$ -th iteration of ISTA at stage  $K$  by the backtracking line search, for all  $t$  and  $K$ . Note that at the beginning for each stage of pathwise optimization (outer loop), the step size parameter is reassigned by some initial value  $L_{[K]}^{(0)} = L_{\max}$ , e.g., by taking  $L_{\max} = \|\nabla^2 \mathcal{L}_{\mu_{[K]}}(0)\|_F$  as an upper bound of the Lipschitz constant of the gradient  $\nabla \mathcal{L}_{\mu_{[K]}}$ .

### 3 Computational and Statistical Analysis

Recall that  $\theta^*$  is the true model parameter in the linear model. For notational convenience, given a real value  $r > 0$ , we denote the neighborhood of  $\theta^*$  with respect to  $r$  as

$$\mathcal{B}_r = \{\theta \in \mathbb{R}^d : \|\theta - \theta^*\|_2^2 \leq r\}.$$

We first define two important notions of locally restricted strong convexity and smoothness for the smooth surrogate loss function  $\mathcal{L}_\mu$  with respect to  $\mathcal{B}_r$ .

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**Algorithm 1:** Pathwise Iterative Smoothing Shrinkage Thresholding Algorithm (PIS<sup>2</sup>TA) for solving the SQRT-Lasso optimization (1.4).  $\widehat{\theta}_{[K]}$  is denoted as the output solution of the  $K$ -th optimization stage corresponding to  $\lambda_{[K]}$  and  $\mu_{[K]}$ ;  $\varepsilon_{[K]}$  is a prespecified precision for the  $K$ -th optimization stage.

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**Input:**  $y, X, N, \lambda_{[N]}, \mu_{[N]}, \varepsilon_{[N]}, L_{\max} > 0$

**Initialize:**  $\widehat{\theta}_{[0]} \leftarrow 0, \lambda_{[0]} \leftarrow \|\nabla \mathcal{L}_{\mu_{[0]}}(0)\|_{\infty}, \mu_{[0]} \leftarrow \|y\|_2, \eta_{\lambda} \leftarrow (\lambda_{[N]}/\lambda_{[0]})^{1/N}, \eta_{\mu} \leftarrow (\mu_{[N]}/\mu_{[0]})^{1/N}$

**For:**  $K = 1, \dots, N$

$$\lambda_{[K]} \leftarrow \eta_{\lambda} \lambda_{[K-1]}, \mu_{[K]} \leftarrow \eta_{\mu} \mu_{[K-1]}, \theta_{[K]}^{(0)} \leftarrow \widehat{\theta}_{[K-1]}, L_{[K]}^{(0)} \leftarrow L_{\max}, \varepsilon_{[K]} \leftarrow \begin{cases} \lambda_{[K]}/4, & \text{if } K < N \\ \varepsilon_{[N]}, & \text{if } K = N \end{cases}$$

$$\widehat{\theta}_{[K]} \leftarrow \text{ISTA}\left(\lambda_{[K]}, \mu_{[K]}, \theta_{[K]}^{(0)}, L_{[K]}^{(0)}, \varepsilon_{[K]}\right)$$

**End For**

**Return:**  $\widehat{\theta}_{[N]}$

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**Algorithm 2:** Iterative Shrinkage and Thresholding Algorithm of PIS<sup>2</sup>TA. (ISTA)

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**Input:**  $\lambda_{[K]}, \mu_{[K]}, \theta_{[K]}^{(0)}, L_{[K]}^{(0)}, \varepsilon_{[K]}$

**Initialize:**  $t \leftarrow 0$

**Repeat:**

$$t \leftarrow t + 1$$

$$L_{[K]}^{(t)} \leftarrow \min\{2\widetilde{L}_{[K]}^{(t)}, L_{\max}\}, \text{ where } \widetilde{L}_{[K]}^{(t)} \leftarrow \text{LineSearch}\left(\lambda_{[K]}, \mu_{[K]}, \theta_{[K]}^{(t-1)}, L_{[K]}^{(t-1)}\right)$$

$$\theta_{[K]}^{(t)} \leftarrow \mathcal{T}_{L_{[K]}^{(t)}, \mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t-1)})$$

**Until:**  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \leq \varepsilon_{[K]}$

**Return:**  $\theta_{[K]}^{(t)}$

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**Algorithm 3:** Backtracking Line search of PIS<sup>2</sup>TA. (LineSearch)

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**Input:**  $\lambda_{[K]}, \mu_{[K]}, \theta_{[K]}^{(t-1)}, L_{[K]}^{(t-1)}$

**Initialize:**  $\widetilde{L}_{[K]}^{(t)} = L_{[K]}^{(t-1)}$

**Repeat:**

$$\theta_{[K]}^{(t)} = \mathcal{T}_{\widetilde{L}_{[K]}^{(t)}, \mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t-1)})$$

**If**  $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) < \mathcal{Q}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}, \theta_{[K]}^{(t-1)})$

$$\widetilde{L}_{[K]}^{(t)} = \widetilde{L}_{[K]}^{(t)}/2$$

**End If**

**Until:**  $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \geq \mathcal{Q}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}, \theta_{[K]}^{(t-1)})$

**Return:**  $\widetilde{L}_{[K]}^{(t)}$

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**Definition 3.1.** For any  $v, w \in \mathcal{B}_r$  with  $\|v - w\|_0 \leq s$ , we say  $\mathcal{L}_\mu$  is *locally restricted strongly convex* (LRSC) and *locally restricted strongly smooth* (LRSS) on  $\mathcal{B}_r$  at sparsity level  $s$ , if there exist universal constants  $\rho_s^-, \rho_s^+ \in (0, \infty)$  such that

$$\frac{\rho_s^-}{2} \|v - w\|_2^2 \leq \mathcal{L}_\mu(v) - \mathcal{L}_\mu(w) - \nabla \mathcal{L}_\mu(w)^\top (v - w) \leq \frac{\rho_s^+}{2} \|v - w\|_2^2, \quad (3.1)$$

We define the locally restricted condition number at sparsity level  $s$  as  $\kappa_s = \rho_s^+ / \rho_s^-$ .

The LRSC and LRSS properties are locally constrained variants of restricted strong convexity and smoothness (Agarwal et al., 2010; Xiao and Zhang, 2013), which are keys to establishing the strong convergence guarantees of our proposed algorithm in high dimensions.

Next, we introduce two key assumptions for establishing our computational theory.

**Assumption 3.2.** The sequence of the regularization parameters satisfies

$$\lambda_{[1]} > \dots > \lambda_{[N]} \geq 6 \|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty.$$

Recall that  $\lambda_{[N]} = \lambda_{\text{SQRT}}$ . Assumption 3.2 requires that  $\lambda_{[N]}$  is large enough such that the irrelevant variables can be eliminated. This is widely used in high dimensional analysis (Bickel et al., 2009; Negahban et al., 2012).

**Assumption 3.3.** For some index  $N_1 < N$ ,  $\mathcal{L}_{\mu_{[K]}}$  satisfies LRSC and LRSS properties on  $\mathcal{B}_r$  for all  $N_1 \leq K \leq N$ , where  $r$  satisfies

$$r \geq s^* \left( \frac{8\lambda_{[N_1]}}{\rho_{s^*+\widetilde{s}}^-} \right)^2.$$

Specifically, (3.1) holds with  $\rho_{s^*+2\widetilde{s}}^+, \rho_{s^*+2\widetilde{s}}^- \in (0, \infty)$ , where  $\widetilde{s} = C_1 s^* > (196\kappa_{s^*+2\widetilde{s}}^2 + 144\kappa_{s^*+2\widetilde{s}})s^*$ ,  $C_1 \in \mathbb{R}^+$  is a constant and  $\kappa_{s^*+2\widetilde{s}} = \rho_{s^*+2\widetilde{s}}^+ / \rho_{s^*+2\widetilde{s}}^-$ .

Assumption 3.3 requires that  $\mathcal{L}_\mu$  satisfies LRSC and LRSS properties when the estimation error satisfies  $\|\theta - \theta^*\|_2^2 \leq r$  and the number of irrelevant coordinates of solutions is bounded by  $\widetilde{s}$ . Similar assumptions are extensively used in sparse learning algorithms (Xiao and Zhang, 2013; Wang et al., 2014).

### 3.1 Computational Theory

Our analysis consists of two phases, Phase I and Phase II, depending on the the distance to the true model parameter  $\theta^*$  and sparsity of the solution  $\theta$  along the path. For notational convenience, given an integer  $s$  with  $0 \leq s \leq d$ , we denote

$$\mathcal{B}_r^s = \mathcal{B}_r \cap \{\theta \in \mathbb{R}^d : \|\theta - \theta^*\|_0 \leq s\}.$$

Let  $N_1 \in \{1, \dots, N\}$  be the cut-off stage between two phases. Phase I corresponds to the first  $N_1$  stages of the pathwise optimization, where we cannot guarantee the estimator enters the sparse



domain  $\mathcal{B}_r^{s^*+\tilde{s}}$ . Thus we only establish a sublinear convergence for Phase I. But Phase I is still computationally efficient, since we can choose reasonably large  $\varepsilon_{[K]}$  to facilitate early stopping and large  $\mu_{[K]}$  for better constant dependence of convergence guarantee, for all  $K = 1, \dots, N_1$ . Phase II corresponds to the subsequent  $(N - N_1)$  stages, where we can guarantee the estimator to be in  $\mathcal{B}_r^{s^*+\tilde{s}}$ . Thus LRSC and LRSS hold, and a linear convergence can be established.

**Theorem 3.4.** Suppose that Assumptions 3.2 and 3.3 hold. At the  $K$ -th optimization stage, let  $\bar{\theta}_{[K]}$  be the unique sparse global optimum to (1.4) and  $\widehat{\theta}_{[K]}$  be the output solution, which satisfies  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K]}) \leq \varepsilon_{[K]}$  for all  $K = 1, \dots, N$ . We denote

$$\mathcal{S}^* = \{j \mid \theta_j^* \neq 0\}, \bar{\mathcal{S}}^* = \{j \mid \theta_j^* = 0\}, \text{ and } s^* = |\mathcal{S}^*|.$$

Recall that  $\eta_\lambda$  and  $\eta_\mu$  are the decaying ratios of the geometrically decreasing regularization sequence and smoothing sequence, i.e.,  $\lambda_{[K]} = \eta_\lambda \lambda_{[K-1]}$  and  $\mu_{[K]} = \eta_\mu \mu_{[K-1]}$ . Given  $\mu_{[0]} = \|y\|_2$ ,  $\lambda_{[0]} = \|\nabla \mathcal{L}_\mu(0)\|_\infty$ ,  $\eta_\lambda = (\lambda_{[N]}/\lambda_{[0]})^{1/N} \in (5/6, 1)$ , and  $\mu_{[N]}$  being sufficiently small, we have  $N_1 \in \{1, 2, \dots, N-1\}$  such that the following results hold:

**Phase I:** Let  $R = \max_{K \leq N_1} \left\{ \sup_{\theta} \|\theta - \bar{\theta}_{[K]}\|_2 : \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta) \leq \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) \right\}$ . At the  $K$ -th stage, where  $K = 1, \dots, N_1$ , we need at most  $T_K = \mathcal{O}\left(\frac{\|X\|_2^2 R}{\varepsilon_{[K]} \mu_{[K]} \sqrt{n}}\right)$  iterations to guarantee

- $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \leq \varepsilon_{[K]}$  and
- $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K]}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) = \mathcal{O}\left(\frac{\|X\|_2^2 R^2}{T_K \mu_{[K]} \sqrt{n}}\right).$

Moreover, for  $K = N_1$  we have  $\widehat{\theta}_{[N_1]} \in \mathcal{B}_r^{s^*+\tilde{s}}$ .

**Phase II:** Let  $\alpha = 1 - \frac{1}{8\kappa_{s^*+2\tilde{s}}} < 1$ . At the  $K$ -th stage, where  $K = N_1 + 1, \dots, N$ , we have  $\|[\theta_{[K]}^{(t)}]_{\bar{\mathcal{S}}^*}\|_0 \leq \tilde{s}$  for all  $t$ , which implies  $\|[\widehat{\theta}_{[K]}]_{\bar{\mathcal{S}}^*}\|_0 \leq \tilde{s}$  and  $\|[\bar{\theta}_{[K]}]_{\bar{\mathcal{S}}^*}\|_0 \leq \tilde{s}$ . Moreover, we need at most  $T_K = \mathcal{O}\left(\kappa_{s^*+2\tilde{s}} \log\left(\frac{\kappa_{s^*+2\tilde{s}}^3 \lambda_{[K]}^2}{\varepsilon_{[K]}^2}\right)\right)$  iterations to guarantee

- $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \leq \varepsilon_{[K]},$
- $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K]}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) = \mathcal{O}\left(\alpha^{T_K} \varepsilon_{[K]} \lambda_{[K]} s^*\right)$  and
- $\|\widehat{\theta}_{[K]} - \bar{\theta}_{[K]}\|_2^2 = \mathcal{O}\left(\alpha^{T_K} \varepsilon_{[K]} \lambda_{[K]} s^*\right).$

Theorem 3.4 guarantees that PIS<sup>2</sup>TA achieves an R-linear convergence to the unique sparse global optimum of (1.4), which is as nearly efficient as PISTA for solving Lasso. However, PIS<sup>2</sup>TA for solving SQRT-Lasso requires much less turning effort than PISTA for solving Lasso, since  $\lambda_{[N]}$  is independent of  $\sigma$ . In addition, PIS<sup>2</sup>TA is much more efficient than other competing algorithm for solving SQRT-Lasso, such as ADMM and SOCP + interior point method.

A geometric interpretation of Theorem 3.4 is provided in Figure 3. Since  $\widehat{\theta}_{[N]}$  is our target output, all but the last stage serve as intermediate processes to facilitate fast convergence to  $\widehat{\theta}_{[K]}$ ,

$K = 1, \dots, N-1$ , which do not require high precision solutions. Thus we choose  $\varepsilon_{[K]} = \lambda_{[K]}/4 \gg \varepsilon_{[N]}$  for  $K = 1, \dots, N-1$  such that both Phase I and Phase II are efficient, except the last stage of Phase II, as shown in Figure 4. Only high precision is required for the  $N$ -th stage, e.g.  $\varepsilon_{[N]} = 10^{-5}$ , as only the last regularization parameter  $\lambda_{[N]}$  is of our interest.

In terms of the geometrically decreasing smoothing parameter  $\mu_{[K]}$ , we choose a large initial value  $\mu_{[0]} = \|y\|_2$  to facilitate better convergence guarantees in Phase I. In our analysis,  $\mu_{[K]} \leq \sqrt{n}\sigma/4$  is a sufficient condition for entering Phase II. When we choose a proper value of  $\lambda_{[N]}$  and a sufficiently small  $\mu_{[N]}$  (almost independent to  $\sigma$ , e.g.,  $\mu_{[N]} = 0.01$  will be  $\ll \sqrt{n}\sigma/4$  in practice), then we can guarantee the existence of Phase II. In other word, the existence of  $N_1$  is guaranteed, which does not need to be specified. Moreover, we have that the smoothed region (deep blue region in Figure 3) does not overlap with the linear convergence region (light orange region in Figure 3) in Phase II. This further implies that (1.3) and (1.4) share the same global optimum in Phase II (Proposition 3.8). Combining the argument above, we have that the total number of iterations for the entire solution path is as most

$$\mathcal{O}\left(\kappa_{s^*+2\bar{s}}(N - N_1) \log\left(\frac{\kappa_{s^*+2\bar{s}} \lambda_{[N]}}{\varepsilon_{[N]}}\right) + \sum_{K=1}^{N_1} \frac{\|X\|_2^2 R}{\lambda_{[K]} \mu_{[K]} \sqrt{n}}\right).$$

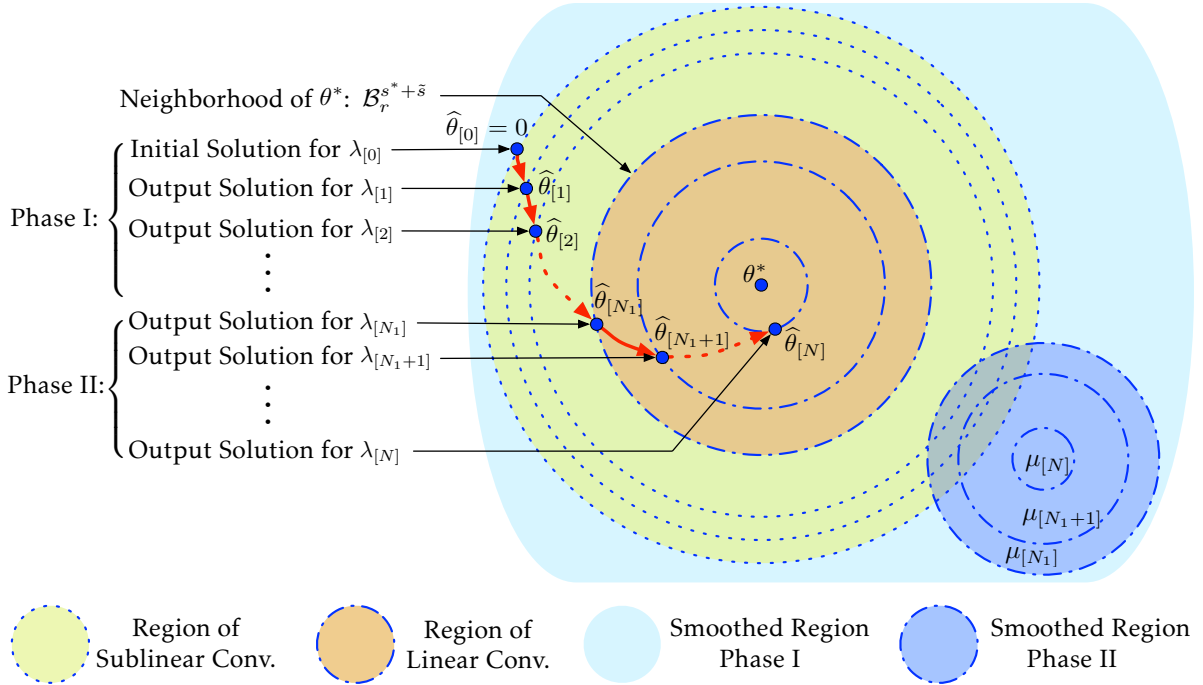


Figure 3: A geometric interpretation of two phases of convergence: sublinear convergence for Phase I (yellow region), and linear convergence for Phase II (orange region). The linear convergence region and the true model parameter  $\theta^*$  do not overlap with the smoothed region.

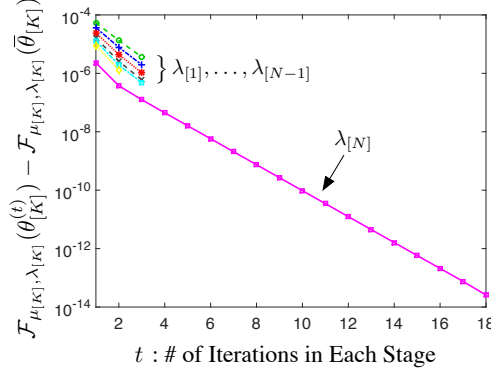


Figure 4: Plots of the objective gaps for all iterations  $t$  of each path following stage (only a few stages are demonstrated for clarity).

### 3.2 Statistical Theory

To analyze the statistical properties of our estimator obtained via PIS<sup>2</sup>TA, we assume that the design matrix  $X$  satisfies the restricted eigenvalue condition as follows.

**Assumption 3.5.** The design matrix  $X$  satisfies the Restricted Eigenvalue (RE) condition, i.e., there exist constants  $\psi_{\min}, \psi_{\max}, \varphi_{\min}, \varphi_{\max} \in (0, \infty)$ , which do not depend on  $(s^*, n, d)$ , such that

$$\psi_{\min} \|v\|_2^2 - \varphi_{\min} \frac{\log d}{n} \|v\|_1^2 \leq \frac{\|Xv\|_2^2}{n} \leq \psi_{\max} \|v\|_2^2 + \varphi_{\max} \frac{\log d}{n} \|v\|_1^2, \quad (3.2)$$

A wide family of examples satisfy the RE condition, such as the correlated sub-Gaussian random design (Rudelson and Zhou, 2013), which has been extensively studied for sparse recovery models (Candes and Tao, 2005; Bickel et al., 2009; Raskutti et al., 2010).

Next, we verify Assumption 3.2 and 3.3 based on the RE condition in the following lemma.

**Lemma 3.6.** Suppose that Assumption 3.5 holds. Given  $\lambda_{[N]} = 24\sqrt{\frac{\log d}{n}}$  and a sufficiently small  $\mu_{[N]}$ , we have  $\lambda_{[N]} \geq 6\|\nabla \mathcal{L}_{\mu_{[N]}}(\theta^*)\|_{\infty}$  with high probability. Moreover, if  $n$  is sufficiently large and  $r = \frac{\sigma^2}{8\psi_{\max}}$ , then  $\mathcal{L}_{\mu_{[K]}}(\theta)$  satisfies LRSC and LRSS properties on  $\mathcal{B}_r$  with high probability for all  $N_1 \leq K \leq N$ . Specifically, (3.1) holds with

$$\rho_{s^*+2\tilde{s}}^+ \leq \frac{8\psi_{\max}}{\sigma} \quad \text{and} \quad \rho_{s^*+2\tilde{s}}^- \geq \frac{\psi_{\min}}{8\sigma},$$

where  $\tilde{s} = C_2 s^* > (196\kappa_{s^*+2\tilde{s}}^2 + 144\kappa_{s^*+2\tilde{s}})s^*$ ,  $C_2 \in \mathbb{R}^+$  is a generic constant, and  $\kappa_{s^*+2\tilde{s}} \leq 64\psi_{\max}/\psi_{\min}$ .

Lemma 3.6 guarantees that Assumption 3.2 holds given properly chosen  $\lambda_{[N]}$  and a sufficiently small  $\mu_{[N]}$ , and Assumption 3.3 holds given the design  $X$  satisfying RE condition, both with high probability. Therefore, by Theorem 3.4, PIS<sup>2</sup>TA achieves an R-linear convergence to the unique sparse global optimum. In the next theorem, we characterize the statistical rate of convergence of PIS<sup>2</sup>TA for SQRT-Lasso.

**Theorem 3.7.** Suppose that Assumption 3.5 holds. For the output solution  $\widehat{\theta}_{[N]}$  obtained from a sufficiently small  $\varepsilon_{[N]}$ ,  $\lambda_{[N]} = 24\sqrt{\frac{\log d}{n}}$ , a sufficiently small  $\mu_{[N]}$ , and a sufficiently large  $n$ , we have

$$\|\widehat{\theta}_{[N]} - \theta^*\|_2 = \mathcal{O}_P\left(\sigma\sqrt{\frac{s^*\log d}{n}}\right) \quad \text{and} \quad \|\widehat{\theta}_{[N]} - \theta^*\|_1 = \mathcal{O}_P\left(\sigma s^*\sqrt{\frac{\log d}{n}}\right).$$

Moreover, let  $\widehat{\sigma} = \frac{\|y - X\widehat{\theta}_{[N]}\|_2}{\sqrt{n}}$  be the estimator of  $\sigma$ . Then we have

$$|\widehat{\sigma} - \sigma| = \mathcal{O}_P\left(\frac{\sigma s^*\log d}{n}\right).$$

Theorem 3.7 guarantees that the output solution  $\widehat{\theta}_{[N]}$  obtained by PIS<sup>2</sup>TA achieves the minimax optimal rate of convergence in parameter estimation (Ye and Zhang, 2010; Raskutti et al., 2011), with the choice of  $\lambda_{[N]}$  being independent of  $\sigma$  and  $\mu_{[N]}$  being almost independent of  $\sigma$  (as long  $\mu_{[N]}$  is sufficiently small). The next proposition shows that (1.3) and (1.4) share the same global optimum, which corresponds to a well fitted model. This implies that neither the unique sparse global optimum  $\bar{\theta}_{[N]}$  nor the linear convergence region falls into the smoothed region, as shown in Figure 3.

**Proposition 3.8.** Under the same assumptions as Theorem 3.7, for all  $\lambda_{[K]}$ 's, where  $K = N_1 + 1, \dots, N$ , (1.3) and (1.4) share the same unique sparse global optimum with high probability.

## 4 Extension to Sparse Precision Matrix Estimation

We consider the TIGER approach proposed in Liu and Wang (2012) for estimating the sparse precision matrix. To be clear, we emphasize that we use  $v_1, v_2, \dots$  (without  $[\cdot]$  for subscript) to index vectors. Let  $X = [x_1^\top, \dots, x_n^\top]^\top \in \mathbb{R}^{n \times d}$  be  $n$  observed data points from a  $d$ -variate Gaussian distribution  $\mathcal{N}_d(0, \Sigma)$ . Our goal is to estimate the sparse precision matrix  $\Theta = \Sigma^{-1}$ . Let the standardized data matrix be

$$Z = X\widehat{\Gamma}^{-1/2} = [z_1^\top, \dots, z_n^\top]^\top \in \mathbb{R}^{n \times d},$$

where  $\widehat{\Gamma} = \text{diag}(\widehat{\Sigma}_{11}, \dots, \widehat{\Sigma}_{dd})$  is a diagonal matrix and  $\widehat{\Sigma} = \frac{1}{n}X^\top X$ . For all  $i = 1, \dots, d$ , we denote

$$z_i = Z_{*\setminus i}\theta_i^* + \widehat{\Gamma}_{ii}^{-1/2}\epsilon_i,$$

where  $\theta_i^* = \widehat{\Gamma}_{ii}^{-1/2}\widehat{\Gamma}_{i\setminus i}^{1/2}(\Sigma_{i\setminus i})^{-1}\Sigma_{\setminus ii} \in \mathbb{R}^{d-1}$  and  $\epsilon_i \sim \mathcal{N}_n(0, \sigma_i^2 I_n)$  with  $\sigma_i^2 = \Sigma_{ii} - \Sigma_{\setminus ii}^\top(\Sigma_{i\setminus i})^{-1}\Sigma_{\setminus ii}$ . Denote  $\tau_i^2 = \sigma_i^2\widehat{\Gamma}_{ii}^{-1}$ , then for  $i = 1, \dots, d$ , we solve

$$\bar{\theta}_i = \underset{\theta_i \in \mathbb{R}^{d-1}}{\text{argmin}} \mathcal{L}_{\mu, i}(\theta_i) + \lambda\|\theta_i\|_1, \quad (4.1)$$

where  $\mathcal{L}_{\mu,i}(\theta_i) = \frac{1}{\sqrt{n}} \|z_i - Z_{*\setminus i} \theta_i\|_\mu$ . Let  $\bar{\tau}_i = \|z_i - Z_{*\setminus i} \bar{\theta}_i\|_\mu$ , then the  $i$ -th column of the precision matrix  $\Theta$  is estimated by

$$\bar{\Theta}_{ii} = \bar{\tau}_i^{-2} \widehat{\Gamma}_{ii}^{-1} \quad \text{and} \quad \bar{\Theta}_{\setminus i i} = -\bar{\tau}_i^{-2} \widehat{\Gamma}_{ii}^{-1/2} \widehat{\Gamma}_{\setminus i \setminus i}^{-1/2} \bar{\theta}_i.$$

Here we solve (4.1) by PIS<sup>2</sup>TA. We first introduce a few mild technical assumptions as follows.

**Assumption 4.1.** Suppose that the true covariance matrix  $\Sigma^*$  and the precision matrix  $\Theta^*$  satisfy:

$$(A1) \quad \Theta^* \in \mathcal{M}(\kappa_\Theta, s^*) = \left\{ \Theta \in \mathbb{R}^{d \times d} : \Theta \succ 0, \Lambda_{\max}(\Theta)/\Lambda_{\min}(\Theta) \leq \kappa_\Theta, \max_i \sum_j \mathbf{1}(\Theta_{ij} \neq 0) \leq s^* \right\},$$

$$(A2) \quad (s^*)^2 \log d = o(n), \text{ and}$$

$$(A3) \quad \limsup_{n \rightarrow \infty} \max_i (\Sigma_{ii}^*)^2 \log d / n < 1/4.$$

We verify the assumptions required by our computational theory by the following lemma.

**Lemma 4.2.** Suppose that Assumption 4.1 holds. Given  $\lambda_{[N]} = 6\sqrt{\frac{5 \log d}{n}}$  and a sufficiently small  $\mu_{[N]}$ , we have  $\lambda_{[N]} \geq 6 \max_{i \in \{1, \dots, d\}} \|\nabla \mathcal{L}_{\mu_{[N]}, i}(\theta_i^*)\|_\infty$  with high probability. Moreover,  $\mathcal{L}_{\mu_{[K]}, i}(\theta_i)$  satisfies LRSC and LRSS properties on  $\mathcal{B}_{r_i}$  for  $r_i = \frac{\sigma_i^2}{12\kappa_\Theta}$  with high probability, for all  $i = 1, \dots, d$  and  $N_1 \leq K \leq N$ . Specifically, for all  $i = 1, \dots, d$ , (3.1) holds with

$$\rho_{s^*+2\tilde{s}}^+ \leq \frac{12\kappa_\Theta}{\sigma_i} \quad \text{and} \quad \rho_{s^*+2\tilde{s}}^- \geq \frac{1}{12\kappa_\Theta \sigma_i},$$

where  $\tilde{s} = C_3 s^* > (196\kappa_{s^*+2\tilde{s}}^2 + 144\kappa_{s^*+2\tilde{s}})s^*$  for a generic constant  $C_3$ , and  $\kappa_{s^*+2\tilde{s}} \leq 144\kappa_\Theta^2$ .

Lemma 4.2 guarantees that Assumption 3.2 and 3.3 hold with high probability given properly chosen  $\lambda_{[N]}$  and a sufficiently small  $\mu_{[N]}$ . Thus, by Theorem 3.4, PIS<sup>2</sup>TA achieves an R-linear convergence to the unique sparse global optimum of (4.1) with high probability for all columns of  $\Theta$ . The next theorem characterizes the statistical rate of convergence of the obtained precision matrix estimator using PIS<sup>2</sup>TA.

**Theorem 4.3.** Suppose that Assumption 4.1 holds. For the output solution  $\widehat{\Theta}_{[N]}$  obtained from a sufficiently small  $\varepsilon_{[N]}$ ,  $\lambda_{[N]} = 6\sqrt{\frac{5 \log d}{n}}$ , and a sufficiently small  $\mu_{[N]}$ , we have

$$\|\widehat{\Theta}_{[N]} - \Theta^*\|_2 = \mathcal{O}_P \left( s^* \|\Theta^*\|_2 \sqrt{\frac{\log d}{n}} \right).$$

Theorem 4.3 implies that our obtained precision matrix estimator attains the minimax optimal rate of convergence in parameter estimation (Liu and Wang, 2012). Moreover, we guarantee that none of the linear convergence region, the output solution  $\widehat{\Theta}_{[N]}$ , or  $\Theta^*$  falls into the smoothed region (Phase II) with high probability. Therefore, PIS<sup>2</sup>TA for estimating the sparse precision matrix shares the same global optimum  $\bar{\Theta}_{[N]}$  with TIGER by Proposition 3.8.

## 5 Numerical Experiments

We investigate the computational and statistical performance of PIS<sup>2</sup>TA through numerical experiments over both synthetic and real data examples. For solving SQRT-Lasso, we compare with ADMM proposed in [Li et al. \(2015\)](#) for computational performance while achieving identical statistical performance. We also compare with PISTA proposed in [Xiao and Zhang \(2013\)](#) for solving Lasso in terms of both computational and statistical evaluations. All simulations are implemented in C with double precision using a PC with an Intel 3.3GHz Core i5 CPU and 16GB memory.

For synthetic data, we generate a training dataset of 200 samples, where each row of the design matrix  $X_{i*}$  is sampled independently from a 2000-dimensional normal distribution  $N(0, \Sigma)$  for all  $i = 1, \dots, 200$ , where  $\Sigma_{jj} = 1$  and  $\Sigma_{jk} = 0.5$  for all  $k \neq j$ . We set  $s^* = 3$  with  $\theta_1^* = 3$ ,  $\theta_2^* = -2$ , and  $\theta_4^* = 1.5$ , and  $\theta_j^* = 0$  for all  $j \neq 1, 2$ , or  $4$ . A validation set of 200 samples for the regularization parameter selection and a testing set of 10,000 samples are also generated to evaluate the prediction accuracy.

We set  $\sigma = 0.5, 1, 2$ , and  $4$  respectively to illustrate the tuning insensitivity. The regularization parameter of both Lasso and SQRT-Lasso is chosen over a geometrically decreasing sequence  $\{\lambda_{[K]}\}_{K=0}^{50}$  with  $\lambda_{50} = \sigma \sqrt{\log d/n}/2$  for Lasso and  $\lambda_{50} = \sqrt{\log d/n}/2$  for SQRT-Lasso. The optimal regularization parameter is determined by  $\lambda_{\text{opt}} = \lambda_{\widehat{N}}$  as  $\widehat{N} = \arg\min_{K \in \{0, \dots, 50\}} \|\widetilde{y} - \widehat{X}\widehat{\theta}_{[K]}\|_2^2$ , where  $\widehat{\theta}_{[K]}$  denotes the obtained estimator using the regularization parameter  $\lambda_{[K]}$ , and  $\widetilde{y}$  and  $\widetilde{X}$  denote the response vector and design matrix of the validation set. For both Lasso and SQRT-Lasso, we set the stopping precision  $\varepsilon_{[K]} = 10^{-5}$  for all  $K = 1, \dots, 50$  to obtain a high precision estimation for each  $\lambda_{[K]}$ . For SQRT-Lasso, we set the smoothing parameter  $\mu_{[K]} = 10^{-3}$  for all  $k = 1, \dots, 50$  for simplicity.

First of all, we compare PIS<sup>2</sup>TA with ADMM proposed in [Li et al. \(2015\)](#)<sup>3</sup> for solving SQRT-Lasso. The backtracking line search described in Algorithm 3 is adopted to accelerate both algorithms. We conduct 500 simulations for all  $\sigma$ 's. The results are presented in Table 1. The PIS<sup>2</sup>TA and ADMM algorithms attain similar objective values, but PIS<sup>2</sup>TA is about 20 times faster than ADMM. Both algorithms also achieve similar estimation errors. Throughout all 500 simulations, we have  $\|y - X\widehat{\theta}_{[\widehat{N}]}\|_2 \gg \mu = 10^{-3}$ . This implies that all obtained optimal estimators are outside the smoothed region of the optimization problem, i.e., the smoothing approach does not hurt the solution accuracy.

Next, we compare the computational and statistical performance between Lasso (solved by PISTA) and SQRT-Lasso (solved by PIS<sup>2</sup>TA). The results averaged over 500 simulations are summarized in Tables 1. In terms of statistical performance, Lasso and SQRT-Lasso attain similar estimation and prediction errors. In terms of computational performance, PIS<sup>2</sup>TA for solving SQRT-Lasso is as efficient as PISTA for solving Lasso, which matches our computational analysis. However, we would like to emphasize that we provide the optimal choice of the regularization parameter, i.e.  $\lambda_{\text{opt}}$ , to both PISTA and PIS<sup>2</sup>TA. In practice that will not be the case and PISTA for solving Lasso would be orders of times slower than PIS<sup>2</sup>TA for solving SQRT-Lasso because of

<sup>3</sup>We do not have any results for the algorithm proposed in [Belloni et al. \(2011\)](#), because it failed to finish 500 simulations in 12 hours. The implementation was based on SDPT3.

parameter tuning.

Moreover, we also examine the optimal regularization parameters for Lasso and SQRT-Lasso. We visualize the distribution of all 500 selected  $\lambda_{\text{opt}}$ 's using the kernel density estimator. Specially, we adopt the Gaussian kernel, and the kernel bandwidth is selected based on the 10-fold cross validation. Figure 5 illustrates the estimated density functions. The horizontal axis corresponds to the rescaled regularization parameter  $\lambda_{\text{opt}}/\sqrt{\log d/n}$ . We see that the optimal regularization parameters of Lasso significantly vary with different values of  $\sigma$ . In contrast, the optimal regularization parameters of SQRT-Lasso are more concentrated. This is consistent with the claimed tuning insensitivity.

Table 1: Quantitative comparison between Lasso (solved by PISTA) and SQRT-Lasso (solved by PIS<sup>2</sup>TA and ADMM) over 500 trials on synthetic data. Mean values and their standard deviations (in the parenthesis) are provided. The estimation error is defined as  $\|\widehat{\theta} - \theta^*\|_2$ . The prediction error is defined as  $\|\widetilde{y} - \widetilde{X}\widehat{\theta}_{[\widehat{N}]}\|_2/\sqrt{n}$ . The residual is defined as  $\|y - X\widehat{\theta}_{[\widehat{N}]}\|_2$  for PIS<sup>2</sup>TA only. PIS<sup>2</sup>TA attains nearly the same estimation and prediction errors as ADMM, but is significantly faster than ADMM over all different settings. In addition, we also observe that all obtained estimators are far outside the smoothed region throughout all 500 trials.

Variance of Noise	Est. Err.		Pred. Err.		Time (in seconds)			Residual
	Lasso	PIS <sup>2</sup> TA	Lasso	PIS <sup>2</sup> TA	Lasso	PIS <sup>2</sup> TA	ADMM	PIS <sup>2</sup> TA
$\sigma = 0.5$	0.2761 (0.0651)	0.2760 (0.0537)	0.5403 (0.0172)	0.5399 (0.0143)	0.8817 (0.2824)	0.9526 (0.2646)	17.260 (3.1723)	5.2505 (0.9591)
$\sigma = 1$	0.5271 (0.1174)	0.5319 (0.1065)	1.0722 (0.0303)	1.0757 (0.0280)	0.9146 (0.3185)	1.0170 (0.2959)	19.762 (4.4387)	10.5209 (1.7760)
$\sigma = 2$	1.0962 (0.2252)	1.1065 (0.2141)	2.1551 (0.0595)	2.1492 (0.0589)	1.1772 (0.4582)	1.1263 (0.4128)	24.406 (4.2786)	20.886 (3.6506)
$\sigma = 4$	2.1275 (0.4247)	2.1356 (0.4033)	4.2928 (0.1112)	4.2963 (0.1079)	1.2913 (0.4855)	1.2544 (0.5074)	26.101 (5.7725)	45.7623 (5.6333)

Finally, we compare PIS<sup>2</sup>TA with ADMM over real data sets for precision matrix estimation. Particularly, we use four real world biology data sets preprocessed by Li and Toh (2010): Estrogen ( $d = 692$ ), Arabidopsis ( $d = 834$ ), Leukemia ( $d = 1,225$ ), and Hereditary ( $d = 1,869$ ). We set three different values for  $\lambda_{[N]}$  such that the obtained estimators achieve different levels of sparse recovery. We set  $N = 50$ , and  $\varepsilon_{[K]} = 10^{-4}$  for all  $K$ 's. The timing performance is summarized in Table 2. As can be seen, PIS<sup>2</sup>TA is 5 to 20 times faster than ADMM on all four data sets<sup>4</sup>.

<sup>4</sup>We do not have any results for the algorithm proposed in Belloni et al. (2011), because it failed to finish the experiments on all four data sets in 12 hours. The implementation was also based on SDPT3.

Table 2: Timing comparison (in seconds) between PIS<sup>2</sup>TA and ADMM on biology data under different levels of sparsity recovery. PIS<sup>2</sup>TA is significantly faster than ADMM over all settings and data sets.

	Estrogen		Arabidopsis		Leukemia		Hereditary	
	PIS <sup>2</sup> TA	ADMM	PIS <sup>2</sup> TA	ADMM	PIS <sup>2</sup> TA	ADMM	PIS <sup>2</sup> TA	ADMM
Sparsity 1%	16.562	175.98	18.404	373.83	30.609	431.45	43.161	498.32
Sparsity 3%	70.622	338.96	81.557	707.52	86.406	812.69	141.65	895.09
Sparsity 10%	188.03	703.24	226.97	1378.1	257.23	1653.1	413.85	1921.6

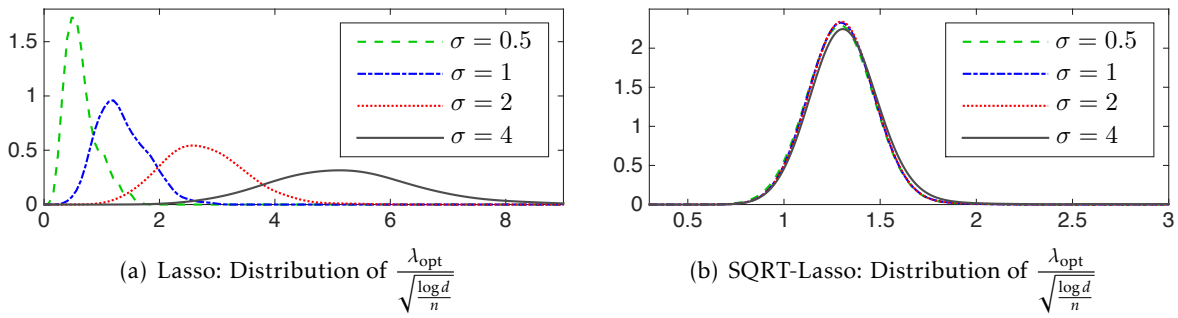


Figure 5: Estimated distributions of  $\frac{\lambda_{\text{opt}}}{\sqrt{\frac{\log d}{n}}}$  over different values of  $\sigma$  for Lasso and SQRT-Lasso.

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## A Intermediate Results of Theorem 3.4 and Theorem 3.7

We introduce some important implications of the proposed assumptions. Recall that  $\mathcal{S}^* = \{j : \theta_j^* \neq 0\}$  be the index set of non-zero entries of  $\theta^*$  with  $s^* = |\mathcal{S}^*|$  and  $\bar{\mathcal{S}}^* = \{j : \theta_j^* = 0\}$  be the complement set. From Lemma 3.6, Assumption 3.3 implies RSC and RSS with parameter  $\rho_{s^*+2\bar{s}}^-$  and  $\rho_{s^*+2\bar{s}}^+$  respectively. By Nesterov (2004), the following conditions are equivalent to RSC and RSS, i.e., for any  $v, w \in \mathbb{R}^d$  satisfying  $\|v - w\|_0 \leq s^* + 2\bar{s}$ , we have

$$\rho_{s^*+2\bar{s}}^- \|v - w\|_2^2 \leq (v - w)^\top \nabla \mathcal{L}_\mu(w) \leq \rho_{s^*+2\bar{s}}^+ \|v - w\|_2^2, \quad (\text{A.1})$$

$$\frac{1}{\rho_{s^*+2\bar{s}}^+} \|\nabla \mathcal{L}_\mu(v) - \nabla \mathcal{L}_\mu(w)\|_2^2 \leq (v - w)^\top \nabla \mathcal{L}_\mu(w) \leq \frac{1}{\rho_{s^*+2\bar{s}}^-} \|\nabla \mathcal{L}_\mu(v) - \nabla \mathcal{L}_\mu(w)\|_2^2. \quad (\text{A.2})$$

From the convexity of  $\ell_1$  norm, we have

$$\|v\|_1 - \|w\|_1 \geq (v - w)^\top g, \quad (\text{A.3})$$

where  $g \in \partial\|w\|_1$ . Combining and (A.1) and (A.3), for any  $v, w \in \mathbb{R}^d$  satisfying  $\|v - w\|_0 \leq s^* + 2\bar{s}$ , we have

$$\mathcal{F}_{\mu,\lambda}(v) - \mathcal{F}_{\mu,\lambda}(w) - (v - w)^\top \nabla \mathcal{F}_{\mu,\lambda}(w) \geq \rho_{s^*+2\bar{s}}^- \|v - w\|_2^2, \quad (\text{A.4})$$

**Remark A.1.** For any  $t$ , the line search satisfies

$$\begin{aligned} \widetilde{L}_{[K]}^{(t)} &\leq L_{[K]}^{(t)} \leq L_{\max}, \quad L_\mu \leq \widetilde{L}_{[K]}^{(t)} \leq L_{[K]}^{(t)} \leq 2L_\mu \quad \text{for all } K = 1, \dots, N, \quad \text{and} \\ \rho_{s^*+2\bar{s}}^+ &\leq \widetilde{L}_{[K]}^{(t)} \leq L_{[K]}^{(t)} \leq 2\rho_{s^*+2\bar{s}}^+ \quad \text{for all } K = N_1 + 1, \dots, N, \end{aligned} \quad (\text{A.5})$$

where  $L_\mu = \min\{L : \|\nabla \mathcal{L}_\mu(v) - \nabla \mathcal{L}_\mu(w)\|_2 \leq L\|x - y\|_2, \forall v, w \in \mathbb{R}^d\}$ .

We first show that when  $\theta$  is sparse and the approximate KKT condition is satisfied, then both estimation error (in  $\ell_2$  norm) and objective error, w.r.t. the true model parameter  $\theta^*$ , are bounded. This characterizes that the initial value  $\theta_{[K]}^{(0)}$  of the  $K$ -th path following stage has desirable statistical properties if we initialize  $\theta_{[K]}^{(0)} = \widehat{\theta}_{[K-1]}$ . This is formalized in Lemma A.2, and its proof is provided in Appendix N.1.

**Lemma A.2.** Suppose that Assumption 3.2 and Assumption 3.3 hold, and  $\lambda \geq \lambda_{[N]}$ . If  $\theta$  satisfies  $\|\theta_{\bar{\mathcal{S}}}\|_0 \leq \bar{s}$  and the approximate KKT condition

$$\min_{g \in \partial\|\theta\|_1} \|\nabla \mathcal{L}_\mu(\theta) + \lambda g\|_\infty \leq \lambda/2, \quad (\text{A.6})$$

then we have

$$\|(\theta - \theta^*)_{\bar{S}^*}\|_1 \leq 5\|(\theta - \theta^*)_{S^*}\|_1, \quad (\text{A.7})$$

$$\|\theta - \theta^*\|_2 \leq \frac{2\lambda\sqrt{s^*}}{\rho_{s^*+2\bar{s}}^-}, \quad (\text{A.8})$$

$$\|\theta - \theta^*\|_1 \leq \frac{12\lambda s^*}{\rho_{s^*+2\bar{s}}^-}, \quad (\text{A.9})$$

$$\mathcal{F}_{\mu,\lambda}(\theta) - \mathcal{F}_{\mu,\lambda}(\theta^*) \leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\bar{s}}^-}. \quad (\text{A.10})$$

Next, we show that if  $\theta$  is sparse and the objective error is bounded, then the estimation error is also bounded. This characterizes that within the  $K$ -th path following stage, good statistical performance is preserved after each proximal-gradient update. This is formalized in Lemma A.3, and its proof is provided in Appendix N.2.

**Lemma A.3.** Suppose that Assumption 3.2 and Assumption 3.3 hold, and  $\lambda \geq \lambda_{[N]}$ . If  $\theta$  satisfies  $\|\theta_{\bar{S}^*}\|_0 \leq \bar{s}$  and the objective satisfies

$$\mathcal{F}_{\mu,\lambda}(\theta) - \mathcal{F}_{\mu,\lambda}(\theta^*) \leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\bar{s}}^-}$$

then we have

$$\|\theta - \theta^*\|_2 \leq \frac{4\lambda\sqrt{3s^*}}{\rho_{s^*+2\bar{s}}^-}, \quad (\text{A.11})$$

$$\|\theta - \theta^*\|_1 \leq \frac{24\lambda s^*}{\rho_{s^*+2\bar{s}}^-}. \quad (\text{A.12})$$

We then show that if  $\theta$  is sparse and the objective error is bounded, then each proximal-gradient update preserves a sparse solution. This indicates that within the  $K$ -th path following stage, each update of  $\theta_{[K]}^{(t)}$  is sparse, thus has good statistical performance. This is formalized in Lemma A.4, and its proof is provided in Appendix N.3.

**Lemma A.4.** Suppose that Assumption 3.2 and Assumption 3.3 hold, and  $\lambda \geq \lambda_{[N]}$ . If  $\theta$  satisfies  $\|\theta_{\bar{S}^*}\|_0 \leq \bar{s}$ ,  $L$  satisfies  $L < 2\rho_{s^*+2\bar{s}}^+$ , and the objective satisfies

$$\mathcal{F}_{\mu,\lambda}(\theta) - \mathcal{F}_{\mu,\lambda}(\theta^*) \leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\bar{s}}^-}$$

then we have

$$\|(\mathcal{I}_{L,\mu,\lambda}(\theta))_{\bar{S}^*}\|_0 \leq \bar{s}. \quad (\text{A.13})$$

Moreover, we show that if  $\theta$  satisfies the approximate KKT condition, then the objective has a bounded error w.r.t. the regularization parameter  $\lambda$ . This allow us to further characterize strong convergence result of the objective error within each path following stage. This is formalized in Lemma A.5, and its proof is provided in Appendix N.4.

**Lemma A.5.** Suppose that Assumption 3.2 and Assumption 3.3 hold, and  $\lambda \geq \lambda_{[N]}$ . If  $\theta$  satisfies

$$\omega_{\mu,\lambda}(\theta) \leq \lambda/2,$$

For any  $\tilde{\lambda} \in [\lambda_{[N]}, \lambda]$ , let  $\bar{\theta} = \operatorname{argmin}_{\theta} \mathcal{F}_{\mu,\tilde{\lambda}}(\theta)$ . Then we have

$$\mathcal{F}_{\mu,\tilde{\lambda}}(\theta) - \mathcal{F}_{\mu,\tilde{\lambda}}(\bar{\theta}) \leq \frac{12(\lambda + \tilde{\lambda}) \left( \omega_{\mu,\lambda}(\theta) + \lambda - \tilde{\lambda} \right) s^*}{\rho_{s^*+2\tilde{s}}^-}.$$

Furthermore, we show that each path following stage has a local linear convergence rate if the initial value  $\theta^{(0)}$  is sparse and satisfies the approximate KKT condition with adequate precision. In addition, the estimation after each proximal-gradient update is also sparse. This is the key result in demonstrating the geometric convergence rate of the algorithm within each path following stage. This is formalized in Lemma A.6, and its proof is provided in Appendix N.5.

**Lemma A.6.** Suppose that Assumption 3.3 holds. If the initialization  $\theta^{(0)}$  for every stage with any  $\lambda$  in Algorithm 1 satisfies

$$\|\theta^{(0)}\|_0 \leq \tilde{s}.$$

Then for any  $t = 1, 2, \dots$ , we have  $\|\theta^{(t)}\|_0 \leq \tilde{s}$ ,

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \leq \left( 1 - \frac{1}{8\kappa_{s^*+2\tilde{s}}} \right)^t \left( \mathcal{F}_{\mu,\lambda}(\theta^{(0)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \right),$$

where  $\bar{\theta} = \operatorname{argmin}_{\theta} \mathcal{F}_{\mu,\lambda}(\theta)$ .

In addition, we provide the sublinear convergence rate when RSC does not hold, based on a refined analysis of the convergence rate for convex objective (not strongly convex) via proximal-gradient method with line search (Nesterov, 2013). Specifically, we provide a sublinear rate of convergence without the need to classify the distance of the initial objective to the optimal objective. This characterizes the convergence behavior when  $\|X(\theta - \theta^*)\|_2$  is large. We formalize this in Lemma A.7, and provide the proof in Appendix N.6.

**Lemma A.7** (Refined result of Theorem 4 in Nesterov (2013)). Given the initialization  $\theta^{(0)}$ , for any  $\theta \in \mathbb{R}^d$  that satisfies  $\mathcal{F}_{\mu,\lambda}(\theta) \leq \mathcal{F}_{\mu,\lambda}(\theta^{(0)})$ , we denote  $R$  as the upper bound of parameter distance, i.e.,

$$\|\theta - \bar{\theta}\|_2 \leq R.$$

Then for any  $t = 1, 2, \dots$ , we have

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \leq \frac{4\|X\|_2^2 R^2}{(t+2)\mu\sqrt{n}}, \quad (\text{A.14})$$

where  $\bar{\theta} = \operatorname{argmin}_{\theta} \mathcal{F}_{\mu,\lambda}(\theta)$ .

Finally, we introduce two results characterizing the proximal-gradient mapping operation, adapted from [Nesterov \(2013\)](#) and [Xiao and Zhang \(2013\)](#) without proof. The first lemma describes a sufficient descent of the objective by proximal-gradient method.

**Lemma A.8** (Adapted from Theorem 2 in [Nesterov \(2013\)](#)). For any  $L > 0$ ,

$$\mathcal{Q}_{\mu,\lambda}(\mathcal{T}_{L,\mu,\lambda}(\theta), \theta) \leq \mathcal{F}_{\mu,\lambda}(\theta) - \frac{L}{2} \|\mathcal{T}_{L,\mu,\lambda}(\theta) - \theta\|_2^2.$$

In addition, if  $\mathcal{L}_\mu(\theta)$  is convex, we have

$$\mathcal{Q}_{\mu,\lambda}(\mathcal{T}_{L,\mu,\lambda}(\theta), \theta) \leq \min_x \mathcal{F}_{\mu,\lambda}(x) + \frac{L}{2} \|x - \theta\|_2^2. \quad (\text{A.15})$$

Further, we have for any  $L \geq L_\mu$ ,

$$\mathcal{F}_{\mu,\lambda}(\mathcal{T}_{L,\mu,\lambda}(\theta)) \leq \mathcal{Q}_{\mu,\lambda}(\mathcal{T}_{L,\mu,\lambda}(\theta), \theta) \leq \mathcal{F}_{\mu,\lambda}(\theta) - \frac{L}{2} \|\mathcal{T}_{L,\mu,\lambda}(\theta) - \theta\|_2^2. \quad (\text{A.16})$$

The next lemma provides an upper bound of the approximate KKT condition parameter  $\omega_{\mu,\lambda}(\cdot)$  defined in (2.4).

**Lemma A.9** (Adapted from Lemma 2 in [Xiao and Zhang \(2013\)](#)). For any  $L > 0$ , if  $L_\mu$  is the Lipschitz constant of  $\nabla \mathcal{L}_\mu$ , then

$$\omega_{\mu,\lambda}(\mathcal{T}_{L,\mu,\lambda}(\theta)) \leq (L + H_L(\theta)) \|\mathcal{T}_{L,\mu,\lambda}(\theta) - \theta\|_2 \leq (L + L_\mu) \|\mathcal{T}_{L,\mu,\lambda}(\theta) - \theta\|_2,$$

where  $H_L(\theta) = \frac{\|\nabla \mathcal{L}_\mu(\mathcal{T}_{L,\mu,\lambda}(\theta)) - \nabla \mathcal{L}_\mu(\theta)\|_2}{\|\mathcal{T}_{L,\mu,\lambda}(\theta) - \theta\|_2}$  is a local Lipschitz constant, which satisfies  $H_L(\theta) \leq L_\mu$ .

## B Proof of Theorem 3.4

We first demonstrate the sublinear rate for initial stages when the estimation error  $\|\theta - \theta^*\|_2$  is large due to large regularization parameter  $\lambda_{[K]}$ . The proof is provided in Appendix F

**Theorem B.1.** For any  $K = 1, \dots, N$  and  $\lambda_{[K]} > 0$ , let  $\bar{\theta}_{[K]} = \operatorname{argmin}_{\theta} \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta)$  be the optimal solution of  $K$ -th stage with regularization parameter  $\lambda_{[K]}$ . For any  $\theta \in \mathbb{R}^d$  that satisfies  $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta) \leq \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)})$ , let  $\|\theta - \bar{\theta}_{[K]}\|_2 \leq R$ . If the initial value  $\theta_{[K]}^{(0)}$  satisfies  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) \leq \lambda_{[K]}/2$ , then within  $K$ -th stage, for any  $t = 1, 2, \dots$ , we have

$$\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) \leq \frac{4\|X\|_2^2 R^2}{(t+2)\mu_{[K]}\sqrt{n}}, \quad (\text{B.1})$$

To achieve the approximate KKT condition  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \leq \varepsilon_{[K]}$ , the number of proximal-gradient steps is no more than

$$\frac{18\|X\|_2^3 R}{\varepsilon_{[K]} \mu_{[K]}^{3/2} n^{3/4} \sqrt{L_{\mu_{[K]}}}} - 3. \quad (\text{B.2})$$

Note that  $L_{\mu_{[K]}} \asymp \|X\|_2^2/(\mu_{[K]}\sqrt{n})$ . Then (B.2) can be simplified as  $\mathcal{O}\left(\frac{\|X\|_2^2 R}{\varepsilon_{[K]}\mu_{[K]}\sqrt{n}}\right)$ .

Next, we demonstrate the linear rate when the estimator satisfies  $\theta \in \mathcal{B}_r^{s^*+\widetilde{s}}$ . The proof is provided in Appendix G.

**Theorem B.2.** Suppose that Assumption 3.2 and Assumption 3.3 hold, and  $\lambda_{[K]} > 0$  for any  $K = N_1, \dots, N$ . Let  $\bar{\theta}_{[K]} = \operatorname{argmin}_{\theta} \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta)$  be the optimal solution of  $K$ -th stage with regularization parameter  $\lambda_{[K]}$ . If the initial value  $\theta_{[K]}^{(0)}$  satisfies  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) \leq \lambda_{[K]}/2$  with  $\|(\theta_{[K]}^{(0)})_{\widetilde{s}^*}\|_0 \leq \widetilde{s}$ , then within  $K$ -th stage, for any  $t = 1, 2, \dots$ , we have

$$\begin{aligned} \|(\theta_{[K]}^{(t)})_{\widetilde{s}^*}\|_0 &\leq \widetilde{s}, \quad \|\theta_{[K]}^{(t)} - \bar{\theta}_{[K]}\|_2^2 \leq \left(1 - \frac{1}{8\kappa_{s^*+2\widetilde{s}}}\right)^t \frac{24\lambda_{[K]}s^*\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)})}{(\rho_{s^*+2\widetilde{s}}^-)^2} \quad \text{and} \\ \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) &\leq \left(1 - \frac{1}{8\kappa_{s^*+2\widetilde{s}}}\right)^t \frac{24\lambda_{[K]}s^*\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)})}{\rho_{s^*+2\widetilde{s}}^-}, \end{aligned} \quad (\text{B.3})$$

- (1) For  $K = N_1, \dots, N-1$ , to achieve the approximate KKT condition  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \leq \lambda_{[K]}/4$ , the number of proximal-gradient steps is no more than

$$\frac{\log\left(1536(1+2\kappa_{s^*+2\widetilde{s}})^2 s^* \kappa_{s^*+2\widetilde{s}}\right)}{\log(8\kappa_{s^*+2\widetilde{s}}/(8\kappa_{s^*+2\widetilde{s}}-1))}. \quad (\text{B.4})$$

- (2) For  $K = N$ , to achieve the approximate KKT condition  $\omega_{\mu_{[N]}, \lambda_{[N]}}(\theta_{[N]}^{(t)}) \leq \varepsilon_{[N]}$ , the number of proximal-gradient steps is no more than

$$\frac{\log\left(96(1+2\kappa_{s^*+2\widetilde{s}})^2 \lambda_{[N]}^2 s^* \kappa_{s^*+2\widetilde{s}}/\varepsilon_{[N]}^2\right)}{\log(8\kappa_{s^*+2\widetilde{s}}/(8\kappa_{s^*+2\widetilde{s}}-1))}. \quad (\text{B.5})$$

From basic inequalities, since  $\kappa_{s^*+2\widetilde{s}} \geq 1$ , we have

$$\log\left(\frac{8\kappa_{s^*+2\widetilde{s}}}{8\kappa_{s^*+2\widetilde{s}}-1}\right) \geq \log\left(1 + \frac{1}{8\kappa_{s^*+2\widetilde{s}}-1}\right) \geq \frac{1}{8\kappa_{s^*+2\widetilde{s}}}.$$

Then (B.4) and (B.5) can be simplified as  $\mathcal{O}\left(\kappa_{s^*+2\widetilde{s}}\left(\log\left(\kappa_{s^*+2\widetilde{s}}^3 s^*\right)\right)\right)$  and  $\mathcal{O}\left(\kappa_{s^*+2\widetilde{s}}\left(\log\left(\kappa_{s^*+2\widetilde{s}}^3 \lambda_{[N]}^2 s^*/\varepsilon_{[N]}^2\right)\right)\right)$  respectively.

As can be seen from Theorem B.2, when the initial value  $\theta_{[K]}^{(0)}$  satisfies the approximate KKT condition  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) \leq \lambda_{[K]}/2$  with  $\theta_{[K]}^{(0)} \in \mathcal{B}_r^{s^*+\widetilde{s}}$ , then we can guarantee the geometric convergence rate of the estimated objective value towards the minimal objective. Next, we show that if the optimal solution  $\widehat{\theta}_{[K-1]}$  from  $K-1$ -th path following stage satisfies the approximate KKT condition and the regularization parameter  $\lambda_{[K]}$  in the  $K$ -th path following stage is chosen properly, then  $\widehat{\theta}_{[K-1]}$  satisfies the approximate KKT condition for  $\lambda_{[K]}$  with a slightly larger bound. This characterizes that good computational properties are preserved by using the warm start  $\theta_{[K]}^{(0)} = \widehat{\theta}_{[K-1]}$  and geometric sequence of regularization parameters  $\lambda_{[K]}$ . We formalize this notion in Lemma B.3, and its proof is provided in Appendix H.

**Lemma B.3.** Let  $\widehat{\theta}_{[K-1]}$  be the approximate solution of  $K-1$ -th path following state, which satisfies the approximate KKT condition  $\omega_{\mu_{[K-1]}, \lambda_{[K-1]}}(\widehat{\theta}_{[K-1]}) \leq \lambda_{[K-1]}/4$ . Then we have

$$\omega_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K-1]}) \leq \lambda_{[K]}/2,$$

where  $\lambda_{[K]} = \eta_\lambda \lambda_{[K-1]}$  with  $\eta_\lambda \in (5/6, 1)$ .

Combining Theorem (B.2) and Lemma (B.3), we can achieve the global convergence in terms of the objective value using the path following proximal-gradient method. We have the bounds of iterations  $T_K$  in Phase 2 directly from (B.4) and (B.5) of Theorem B.2.

Finally, we obtain the objective gap of  $N$ -th stage via analogous argument. Specifically, in the  $N$ -th (final) path following stage, when the number of iterations for proximal method is large enough such that  $\omega_{\mu_{[N]}, \lambda_{[N]}}(\theta_{[N]}^{(t)}) \leq \varepsilon_{[N]}$  holds, then we obtain the result from Lemma A.5 with  $\lambda = \widetilde{\lambda} = \lambda_{[N]}$ .

Finally, we need to show that there exists some  $N_1 \in \{1, \dots, N\}$  such that  $\widehat{\theta}_{[N_1]} \in \mathcal{B}_r^{s^* + \widetilde{s}}$ . We demonstrate this result in Lemma B.4 and provide its proof in Appendix I.

**Lemma B.4.** Suppose that Assumption 3.2 and Assumption 3.3 holds, and the approximate KKT satisfies  $\omega_{\mu, \lambda}(\theta) \leq \lambda/4$ . If  $\mu \leq \sqrt{n}\sigma/4$  and  $\frac{\rho_{s^* + \widetilde{s}}}{8} \sqrt{\frac{T}{s^*}} > \lambda > \lambda_{[N]}$ , then we have

$$\|\theta - \theta^*\|_2^2 \leq r \quad \text{and} \quad \|\theta_{\widetilde{s}^*}\|_0 \leq \widetilde{s}.$$

Lemma B.4 guarantees that there exists some  $N_1 < N$  such that for  $\lambda_{[N_1]} > \lambda_{[N]}$ , the approximate solution  $\widehat{\theta}_{[N_1]}$  satisfies the approximate KKT condition and  $\widehat{\theta}_{[N_1]} \in \mathcal{B}_r^{s^* + \widetilde{s}}$ , then we enter Phase 2 that has a strong linear convergence by Theorem B.2. Thus we finish the proof.

We further provide a bound of the total number of proximal-gradient steps for each  $\lambda_{[K]}$  in the following lemma for interested readers. The proof is provided in Appendix J.

**Lemma B.5.** For each  $\lambda_{[K]}$ ,  $K = 1, \dots, N$ , if we restart the line search with a large enough  $L_{\max}$ , then the total number of proximal-gradient steps is no more than  $2(T_K + 1) + \max\{(\log_2 L_{\max} - \log_2 \rho_{s^* + 2\widetilde{s}}^+), 0\}$ .

## C Proof of Lemma 3.6

We divide the proof into two intermediate lemmas. We first show that Assumption 3.2 holds with properly choose  $\mu_{[N]}$  and  $\lambda_{[N]}$ , conditioning on the sub-Gaussianity of  $\epsilon$ .

**Lemma C.1.** Given  $\mu_{[N]} \leq \frac{\sqrt{n}\sigma}{4}$  and  $\lambda_{[N]} = 24\sqrt{\frac{\log d}{n}}$ , then  $\lambda_{[N]} \geq 6\|\nabla \mathcal{L}_{\mu_{[N]}}(\theta^*)\|_\infty$  with probability at least  $1 - 2d^{-1} - \exp(-\frac{n}{32})$ .

*Proof.* By  $y = X\theta^* + \epsilon$  and (C.4), we have

$$\nabla \mathcal{L}_{\mu_{[N]}}(\theta^*) = \frac{X^\top (X\theta^* - y)}{\max\{\sqrt{n}\mu_{[N]}, \sqrt{n}\|y - X\theta^*\|_2\}} = -\frac{X^\top \epsilon}{\max\{\sqrt{n}\mu_{[N]}, \sqrt{n}\|\epsilon\|_2\}}. \quad (\text{C.1})$$



Since  $\epsilon$  has i.i.d. sub-Gaussian entries and  $\mathbb{E}[\epsilon_i] = 0$  and  $\mathbb{E}[\epsilon_i^2] = \sigma^2$  for all  $i = 1, \dots, n$ , then we have from [Wainwright \(2015\)](#) that

$$\mathbb{P}\left[\|\epsilon\|_2^2 \leq \frac{1}{4}n\sigma^2\right] \leq \exp\left(-\frac{n}{32}\right), \quad (\text{C.2})$$

From [Negahban et al. \(2012\)](#), we have the following result.

**Lemma C.2.** Assume  $X$  satisfies  $\|x_j\|_2 \leq \sqrt{n}$  for all  $j = 1, \dots, d$  and  $\epsilon$  has i.i.d. zero-mean sub-Gaussian entries with  $\mathbb{E}[w_i^2] = \sigma^2$  for all  $i = 1, \dots, n$ , then we have

$$\mathbb{P}\left[\frac{1}{n}\|X^\top \epsilon\|_\infty \geq 2\sigma\sqrt{\frac{\log d}{n}}\right] \leq 2d^{-1}.$$

Combining (C.1), (C.2), and Lemma C.2, then with probability at least  $1 - 2d^{-1} - \exp\left(-\frac{n}{32}\right)$ , we have

$$\|\nabla \mathcal{L}_{\mu_{[N]}}(\theta^*)\|_\infty \leq \frac{4\sqrt{\log d/n}}{\max\{\sqrt{2}\mu_{[N]}/(\sqrt{n}\sigma), 1\}}.$$

Then Assumption 3.2 holds when  $\mu_{[N]} \leq \frac{\sqrt{n}\sigma}{4}$ . □

Next, we show that Assumption 3.3 holds with desired parameters when  $r = \frac{\sigma^2}{8\psi_{\max}}$ ,  $\mu_{[N]}$  is sufficiently small, and Assumption 3.5 holds.

**Lemma C.3.** Suppose that Assumption 3.5 holds. Given  $r = \frac{\sigma^2}{8\psi_{\max}}$   $\mu \ll \frac{\sqrt{n}\sigma}{4}$  is sufficiently small, then with probability at least  $1 - 7\exp\left(-\frac{n}{144}\right)$ , we have that  $\mathcal{L}_{\mu_{[K]}}(\theta)$  satisfies LRSC and LRSS properties on  $\mathcal{B}_r^{s^*+2\tilde{s}}$  for all  $N_1 \leq K \leq N$ .

*Proof.* It is easy to see that we can always choose a sufficiently small  $\mu_{[N]}$  such that for all  $K \geq N_1$ ,  $\mu_{[K]} \leq \frac{\sqrt{n}\sigma}{4}$ . For notational simplicity, we ignore the index of  $K$  for  $\mu$  for the rest of this proof. We divide the proof into two steps.

**Step 1.** When  $X$  satisfies the RE condition (3.2), i.e.

$$\psi_{\min}\|v\|_2^2 - \varphi_{\min}\frac{\log d}{n}\|v\|_1^2 \leq \frac{\|Xv\|_2^2}{n} \leq \psi_{\max}\|v\|_2^2 + \varphi_{\max}\frac{\log d}{n}\|v\|_1^2,$$

Denote  $s = s^* + 2\tilde{s}$ . Since  $\|v\|_0 \leq s$ , which implies  $\|v\|_1^2 \leq s\|v\|_2^2$ , then we have

$$\left(\psi_{\min} - \varphi_{\min}\frac{s\log d}{n}\right)\|v\|_2^2 \leq \frac{\|Xv\|_2^2}{n} \leq \left(\psi_{\max} + \varphi_{\max}\frac{s\log d}{n}\right)\|v\|_2^2,$$

Then there exists a universal constant  $c_1$  such that if  $n \geq c_1 s^* \log d$ , we have

$$\frac{1}{2}\psi_{\min}\|v\|_2^2 \leq \frac{\|Xv\|_2^2}{n} \leq 2\psi_{\max}\|v\|_2^2. \quad (\text{C.3})$$

**Step 2.** Conditioning on (C.3), we show that  $\mathcal{L}_\mu$  satisfies LRSC and LRSS with high probability.

The gradient of  $\mathcal{L}_\mu(\theta)$  is

$$\nabla \mathcal{L}_\mu(\theta) = \frac{1}{\sqrt{n}} \left( \left( \frac{\partial \|y - X\theta\|_\mu}{\partial (y - X\theta)} \right)^\top \left( \frac{\partial (y - X\theta)}{\partial \theta} \right)^\top \right)^\top = \frac{X^\top (X\theta - y)}{\max\{\sqrt{n}\mu, \sqrt{n}\|y - X\theta\|_2\}}. \quad (\text{C.4})$$

The Hessian of  $\mathcal{L}_\mu(\theta)$  is

$$\nabla^2 \mathcal{L}_\mu(\theta) = \frac{1}{n} \frac{\partial (-X^\top \tilde{z})}{\partial \theta} = \begin{cases} \frac{X^\top X}{\sqrt{n}\mu}, & \text{if } \|y - X\theta\|_2 < \mu \\ \frac{1}{\sqrt{n}\|y - X\theta\|_2} X^\top \left( I - \frac{(y - X\theta)(y - X\theta)^\top}{\|y - X\theta\|_2^2} \right) X, & \text{o.w.} \end{cases} \quad (\text{C.5})$$

For notational convenience, we define  $\Delta = v - w$  for any  $v, w \in \mathcal{B}_s^*$ . Also denote the residual of the first order Taylor expansion as

$$\delta \mathcal{L}_\mu(w + \Delta, w) = \mathcal{L}_\mu(w + \Delta) - \mathcal{L}_\mu(w) - \nabla \mathcal{L}_\mu(w)^\top \Delta.$$

Using the first order Taylor expansion of  $\mathcal{L}_\mu$  at  $w$  and the Hessian of  $\mathcal{L}_\mu$  in (C.5), we have from mean value theorem that there exists some  $\alpha \in [0, 1]$  such that

$$\delta \mathcal{L}_\mu(w + \Delta, w) = \begin{cases} \frac{\Delta^\top X^\top X \Delta}{\sqrt{n}\mu}, & \text{if } \|xi\|_2 < \mu \\ \frac{1}{\sqrt{n}\|xi\|_2} \Delta^\top X^\top \left( I - \frac{xi xi^\top}{\|xi\|_2^2} \right) X \Delta, & \text{o.w.} \end{cases} \quad (\text{C.6})$$

where  $xi = y - X(w + \alpha\Delta)$ . For notational simplicity, let's denote  $\dot{z} = X(v - \theta^*)$  and  $\ddot{z} = X(w - \theta^*)$ , which can be considered as two fixed vectors in  $\mathbb{R}^n$ . Without loss of generality, assume  $\|\dot{z}\|_2 \leq \|\ddot{z}\|_2$ . Then we have

$$\|\dot{z}\|_2^2 \leq \|\ddot{z}\|_2^2 \stackrel{(i)}{\leq} 2\psi_{\max} n \|w - \theta^*\|_2^2 \stackrel{(ii)}{\leq} \frac{n\sigma^2}{4},$$

where (i) is from (C.3) and (ii) is from  $\|w - \theta^*\|_2^2 \leq r = \frac{\sigma^2}{8\psi_{\max}}$ . Further, we have

$$xi = y - X(w + \alpha\Delta) = \epsilon - X(w + \alpha\Delta - \theta^*) = \epsilon - \alpha\dot{z} - (1 - \alpha)\ddot{z}, \text{ and } X\Delta = \dot{z} - \ddot{z}.$$

We have from Wainwright (2015) that

$$\mathbb{P}[\|\epsilon\|_2^2 \leq n\sigma^2(1 - \delta)] \leq \exp\left(-\frac{n\delta^2}{16}\right), \quad (\text{C.7})$$

Then by taking  $\delta = 1/3$  in (C.7), we have with probability  $1 - \exp(-\frac{n}{144})$ ,

$$\|xi\|_2 \geq \|\epsilon\|_2 - \alpha\|\dot{z}\|_2 - (1 - \alpha)\|\ddot{z}\|_2 \geq \|\epsilon\|_2 - \|\ddot{z}\|_2 \geq \frac{4}{5}\sqrt{n}\sigma - \frac{1}{2}\sqrt{n}\sigma \geq \frac{1}{4}\sqrt{n}\sigma. \quad (\text{C.8})$$

We first discuss the RSS property. From (C.8), we have  $\|xi\|_2 \geq \mu$ , then we have from (C.6) that

$$\begin{aligned} \delta \mathcal{L}_\mu(w + \Delta, w) &= \frac{1}{\sqrt{n}\|xi\|_2} \Delta^\top X^\top \left( I - \frac{xi xi^\top}{\|xi\|_2^2} \right) X \Delta = \frac{1}{\sqrt{n}\|xi\|_2} \left( \|X\Delta\|_2^2 - \frac{(xi^\top X\Delta)^2}{\|xi\|_2^2} \right) \\ &\leq \frac{\|X\Delta\|_2^2}{\sqrt{n}\|xi\|_2} \leq \frac{8\psi_{\max}}{\sigma} \|\Delta\|_2^2, \end{aligned} \quad (\text{C.9})$$

where the last inequality is from (C.3) and (C.8).

Next, we verify the RSC property. From (C.8), we have  $\|xi\|_2 \geq \mu$ . We want to show that with high probability, there exists some constant  $a \in (0, 1)$  such that

$$\left| \frac{xi^\top}{\|xi\|_2} X\Delta \right| \leq \sqrt{1-a} \|X\Delta\|_2. \quad (\text{C.10})$$

Consequently, we have

$$\Delta^\top X^\top \left( I - \frac{xi xi^\top}{\|xi\|_2^2} \right) X\Delta = \|X\Delta\|_2^2 - \left( \frac{xi^\top}{\|xi\|_2} X\Delta \right)^2 \geq a \|X\Delta\|_2^2.$$

This further implies

$$\delta \mathcal{L}_\mu(w + \Delta, w) = \frac{1}{\sqrt{n} \|xi\|_2} \Delta^\top X^\top \left( I - \frac{xi xi^\top}{\|xi\|_2^2} \right) X\Delta \geq \frac{a \|X\Delta\|_2^2}{\sqrt{n} \|xi\|_2} \geq \frac{a \psi_{\min}}{2 \|xi\|_2 / \sqrt{n}} \|\Delta\|_2^2, \quad (\text{C.11})$$

where the last inequality is from (C.3).

The remaining part is to prove (C.10). Since  $\|\dot{z}\|_2 \leq \|\ddot{z}\|_2$ , then for any real constant  $a \in (0, 1)$ ,

$$\begin{aligned} & \mathbb{P} \left[ \left| \frac{xi^\top}{\|xi\|_2} X\Delta \right| \leq \sqrt{1-a} \|X\Delta\|_2 \right] \\ &= \mathbb{P} \left[ \left| \frac{(\epsilon - \alpha \dot{z} - (1-\alpha)\ddot{z})^\top}{\|\epsilon - \alpha \dot{z} - (1-\alpha)\ddot{z}\|_2} (\dot{z} - \ddot{z}) \right| \leq \sqrt{1-a} \|\dot{z} - \ddot{z}\|_2 \right] \\ &\stackrel{(i)}{\geq} \mathbb{P} \left[ \left| \frac{(\epsilon - \dot{z})^\top (\dot{z} - \ddot{z})}{\|\epsilon - \dot{z}\|_2} \right| \leq \sqrt{1-a} \|\dot{z} - \ddot{z}\|_2 \right] \\ &= \mathbb{P} \left[ \left( \epsilon^\top (\dot{z} - \ddot{z}) - \dot{z}^\top (\dot{z} - \ddot{z}) \right)^2 \leq (1-a) \|\epsilon - \dot{z}\|_2^2 \|\dot{z} - \ddot{z}\|_2^2 \right] \\ &\stackrel{(ii)}{=} \mathbb{P} \left[ \left( \frac{\epsilon^\top (\dot{z} - \ddot{z})}{\|\dot{z} - \ddot{z}\|_2} \right)^2 + \|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z} \leq (1-a)(\|\epsilon\|_2^2 + \|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z}) \right], \end{aligned} \quad (\text{C.12})$$

where (ii) is from dividing both sides by  $\|\dot{z}\|_2^2$ , and (i) is from a geometric inspection and the randomness of  $\epsilon$ , i.e., for any  $\alpha \in [0, 1]$  and  $\|\dot{z}\|_2 \leq \|\ddot{z}\|_2$ ,

$$\left| \frac{-\dot{z}^\top}{\|\dot{z} - \ddot{z}\|_2} (\dot{z} - \ddot{z}) \right| \leq \left| \frac{(-\alpha \dot{z} - (1-\alpha)\ddot{z})^\top}{\|-\alpha \dot{z} - (1-\alpha)\ddot{z}\|_2} (\dot{z} - \ddot{z}) \right|.$$

The random vector  $\epsilon$  with i.i.d. entries does not affect the inequality above. Let's first discuss one side of the probability in (C.12), i.e.,

$$\begin{aligned} & \mathbb{P} \left[ \left( \frac{\epsilon^\top (\dot{z} - \ddot{z})}{\|\dot{z} - \ddot{z}\|_2} \right)^2 + \|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z} \leq (1-a)(\|\epsilon\|_2^2 + \|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z}) \right] \\ &= \mathbb{P} \left[ (1-a)\|\epsilon\|_2^2 \geq \left( \frac{\epsilon^\top (\dot{z} - \ddot{z})}{\|\dot{z} - \ddot{z}\|_2} \right)^2 + a(\|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z}) \right]. \end{aligned} \quad (\text{C.13})$$

Since  $\epsilon$  has i.i.d. sub-Gaussian entries with  $\mathbb{E}[\epsilon_i] = 0$  and  $\mathbb{E}[\epsilon_i^2] = \sigma^2$  for all  $i = 1, \dots, n$ , then  $\frac{\epsilon^\top(\dot{z}-\ddot{z})}{\|\dot{z}-\ddot{z}\|_2}$  and  $\epsilon^\top \dot{z}$  are also zero-mean sub-Gaussians with variances  $\sigma^2$  and  $\sigma^2 \|\dot{z}\|_2^2$  respectively. We have from [Wainwright \(2015\)](#) that

$$\mathbb{P}\left[\|\epsilon\|_2^2 \leq n\sigma^2(1-\delta)\right] \leq \exp\left(-\frac{n\delta^2}{16}\right), \quad (\text{C.14})$$

$$\mathbb{P}\left[\left(\frac{\epsilon^\top(\dot{z}-\ddot{z})}{\|\dot{z}-\ddot{z}\|_2}\right)^2 \geq n\sigma^2\delta^2\right] \leq \exp\left(-\frac{n\delta^2}{2}\right), \quad (\text{C.15})$$

$$\mathbb{P}\left[\epsilon^\top \dot{z} \leq -n\sigma^2\delta\right] \leq \exp\left(-\frac{n^2\sigma^2\delta^2}{2\|\dot{z}\|_2^2}\right). \quad (\text{C.16})$$

Combining (C.14) – (C.16) with  $\|\dot{z}\|_2^2 \leq n\sigma^2/4$ , we have from union bound that with probability at least  $1 - \exp(-\frac{n}{144}) - \exp(-\frac{n}{128}) - \exp(-\frac{n}{128}) \geq 1 - 3\exp(-\frac{n}{144})$ ,

$$\|\epsilon\|_2^2 \geq \frac{2}{3}n\sigma^2, \quad \left(\frac{\epsilon^\top(\dot{z}-\ddot{z})}{\|\dot{z}-\ddot{z}\|_2}\right)^2 \leq \frac{1}{64}n\sigma^2, \quad -\epsilon^\top \dot{z} \leq \frac{1}{16}n\sigma^2.$$

Combining this with (C.12) and (C.13), for  $a \leq 3/5$ , we have

$$\mathbb{P}\left[\frac{xi^\top}{\|xi\|_2} X\Delta \leq \sqrt{1-a}\|X\Delta\|_2\right] \geq 1 - 3\exp\left(-\frac{n}{144}\right). \quad (\text{C.17})$$

For the other side of the probability in (C.12), we have

$$\begin{aligned} & \mathbb{P}\left[\left(\frac{\epsilon^\top(\dot{z}-\ddot{z})}{\|\dot{z}-\ddot{z}\|_2}\right)^2 + \|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z} \geq -(1-a)(\|\epsilon\|_2^2 + \|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z})\right] \\ &= \mathbb{P}\left[(1-a)\|\epsilon\|_2^2 \geq -\left(\frac{\epsilon^\top(\dot{z}-\ddot{z})}{\|\dot{z}-\ddot{z}\|_2}\right)^2 - (2-a)(\|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z})\right] \\ &\geq \mathbb{P}\left[(1-a)\|\epsilon\|_2^2 \geq \left(\frac{\epsilon^\top(\dot{z}-\ddot{z})}{\|\dot{z}-\ddot{z}\|_2}\right)^2 + a(\|\dot{z}\|_2^2 - 2\epsilon^\top \dot{z})\right]. \end{aligned} \quad (\text{C.18})$$

Combining (C.12), (C.17), and (C.18), we have that (C.10) holds with high probability, i.e., for any  $a \in (0, 3/5)$ ,

$$\mathbb{P}\left[\left|\frac{xi^\top}{\|xi\|_2} X\Delta\right| \leq \sqrt{1-a}\|X\Delta\|_2\right] \geq 1 - 6\exp\left(-\frac{n}{144}\right). \quad (\text{C.19})$$

Now we bound  $\|xi\|_2$  to obtain the desired result. From [Wainwright \(2015\)](#), we have

$$\mathbb{P}\left[\|\epsilon\|_2^2 \geq n\sigma^2(1+\delta)\right] \leq \exp\left(-\frac{n\delta^2}{18}\right) = \exp\left(-\frac{n}{72}\right), \quad (\text{C.20})$$

where the last equality is from taking  $\delta = 1/2$ . From  $xi = \epsilon - \alpha\dot{z} - (1-\alpha)\ddot{z}$ , we have

$$\|xi\|_2 \leq \|\epsilon\|_2 + \alpha\|\dot{z}\|_2 + (1-\alpha)\|\ddot{z}\|_2 \stackrel{(i)}{\leq} \|\epsilon\|_2 + \|\dot{z}\|_2 \stackrel{(ii)}{\leq} \sqrt{\frac{3n}{2}}\sigma + \frac{1}{2}\sqrt{n}\sigma \leq 2\sqrt{n}\alpha. \quad (\text{C.21})$$

where (i) is from  $\|\dot{z}\|_2 \leq \|\ddot{z}\|_2$  and (ii) is from (C.20) and  $\|\dot{z}\|_2^2 \leq n\sigma^2/4$ .

Combining (C.11), (C.19), and (C.21), and setting  $a = 1/2$ , with probability at least  $1 - 7\exp(-\frac{n}{144})$ , we have

$$\delta\mathcal{L}_\mu(w + \Delta, w) \geq \frac{\psi_{\min}}{8\sigma} \|\Delta\|_2^2. \quad (\text{C.22})$$

Combining (C.9) and (C.22), we have LRSC and LRSS hold for  $\mathcal{L}_\mu$  with  $\rho_{s^*+2\bar{s}}^+ \leq \frac{8\psi_{\max}}{\sigma}$  and  $\rho_{s^*+2\bar{s}}^- \geq \frac{\psi_{\min}}{8\sigma}$ , with probability at least  $1 - 7\exp(-\frac{n}{144})$ .

Moreover, we have

$$r = \frac{\sigma^2}{8\psi_{\max}} \stackrel{(i)}{>} s^* \left( 64\sigma \lambda_{[N_1]}/\psi_{\min} \right)^2 \stackrel{(ii)}{\geq} s^* \left( 8\lambda_{[N_1]}/\rho_{s^*+2\bar{s}}^- \right)^2,$$

where (i) is from  $n \geq c_1 s^* \log d$  with a sufficiently large constant  $c_1$  and  $\lambda_{[N_1]} \geq 2\lambda_{[N]} = 48\sqrt{\log d/n}$ , and (ii) is from  $\rho_{s^*+2\bar{s}}^- \geq \frac{\psi_{\min}}{8\sigma}$ . The choice of the constant “2” in  $\lambda_{[N_1]} \geq 2\lambda_{[N]}$  is somewhat arbitrary, which can be any fixed constant larger than  $1/\eta_\lambda$  such that the existence of  $\lambda_{[N_1]}$  is guaranteed.  $\square$

We finish the proof of Lemma 3.6 by combining Lemma C.1 and Lemma C.3.

## D Proof of Theorem 3.7

**Part 1.** We first show that estimation errors are as claimed. Since  $\widehat{\theta}_{[K]}$  is the approximate solution of  $K$ -th path following stage, it satisfies  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K]}) \leq \lambda_{[K]}/4 \leq \lambda_{[K+1]}/2$  for  $t \in [N_1 + 1, N - 1]$ , then we have from Lemma B.3 that

$$\omega_{\mu_{[K+1]}, \lambda_{[K+1]}}(\widehat{\theta}_{[K+1]}^{(0)}) \leq \lambda_{[K+1]}/2.$$

By Theorem B.2, we have for any  $t = 1, 2, \dots$ ,

$$\|(\theta_{[K+1]}^{(t)})_{\bar{S}}\|_0 \leq \bar{s}.$$

Applying Lemma A.2 recursively, we have

$$\|\widehat{\theta}_{[N]} - \theta^*\|_2 \leq \frac{2\lambda_{[N]}\sqrt{s^*}}{\rho_{s^*+2\bar{s}}^-} \quad \text{and} \quad \|\widehat{\theta}_{[N]} - \theta^*\|_1 \leq \frac{12\lambda_{[N]}s^*}{\rho_{s^*+2\bar{s}}^-}.$$

Applying Lemma 3.6 with  $\lambda_{[N]} = 24\sqrt{\log d/n}$  and  $\rho_{s^*+2\bar{s}}^- = \frac{\psi_{\min}}{8\sigma}$ , then by union bound, with probability at least  $1 - 8\exp(-\frac{n}{144}) - 2d^{-1}$ , we have

$$\begin{aligned} \|\widehat{\theta}_{[N]} - \theta^*\|_2 &\leq \frac{384\sigma\sqrt{s^*\log d/n}}{\psi_{\min}}, \\ \|\widehat{\theta}_{[N]} - \theta^*\|_1 &\leq \frac{2304\sigma s^*\sqrt{\log d/n}}{\psi_{\min}}. \end{aligned}$$

**Part 2.** Next, we demonstrate the result of the estimation of variance. Let  $\bar{\theta}_{[N]} = \operatorname{argmin}_{\theta} \mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\theta)$  be the optimal solution of  $K$ -th stage. Apply the argument in Part 1 recursively for  $t \rightarrow \infty$ , we have

$$\|\bar{\theta}_{[N]} - \theta^*\|_1 \leq \frac{2304\sigma s^* \sqrt{\log d/n}}{\psi_{\min}}. \quad (\text{D.1})$$

Denote  $c_1, c_2, \dots$  as positive universal constants. Then we have

$$\begin{aligned} \mathcal{L}_{\mu_{[N]}}(\bar{\theta}_{[N]}) - \mathcal{L}_{\mu_{[N]}}(\theta^*) &\leq \lambda_{[N]}(\|\theta^*\|_1 - \|\bar{\theta}_{[N]}\|_1) \leq \lambda_{[N]}(\|\theta_{\mathcal{S}^*}^*\|_1 - \|(\bar{\theta}_{[N]})_{\mathcal{S}^*}\|_1 - \|(\bar{\theta}_{[N]})_{\mathcal{S}^c}^*\|_1) \\ &\leq \lambda_{[N]}\|(\bar{\theta}_{[N]} - \theta^*)_{\mathcal{S}^*}\|_1 \leq \lambda_{[N]}\|\bar{\theta}_{[N]} - \theta^*\|_1 \stackrel{(ii)}{\leq} c_1 \frac{\sigma s^* \log d}{n}, \end{aligned} \quad (\text{D.2})$$

where (i) is from the value of  $\lambda_{[N]}$  and  $\ell_1$  error bound in (D.1).

On the other hand, from the convexity of  $\mathcal{L}_{\mu_{[N]}}(\theta)$ , we have

$$\begin{aligned} \mathcal{L}_{\mu_{[N]}}(\bar{\theta}_{[N]}) - \mathcal{L}_{\mu_{[N]}}(\theta^*) &\geq (\bar{\theta}_{[N]} - \theta^*)^\top \nabla \mathcal{L}_{\mu_{[N]}}(\theta^*) \geq -\|\nabla \mathcal{L}_{\mu_{[N]}}(\theta^*)\|_\infty \|\bar{\theta}_{[N]} - \theta^*\|_1 \\ &\stackrel{(i)}{\geq} -c_2 \lambda_{[N]} \|\bar{\theta}_{[N]} - \theta^*\|_1 \stackrel{(ii)}{\geq} -c_3 \frac{\sigma s^* \log d}{n}, \end{aligned} \quad (\text{D.3})$$

where (i) is from Assumption 3.2 and (ii) value of  $\lambda_{[N]}$  and  $\ell_1$  error bound in (D.1).

For our choice of  $\mu_{[N]}$  and  $n$ , we have  $\mathcal{L}_{\mu_{[N]}}(\theta) = \frac{1}{\sqrt{n}}\|y - X\theta\|_2 - \frac{\mu_{[N]}}{2}$  by Proposition 3.8, then

$$\mathcal{L}_{\mu_{[N]}}(\bar{\theta}_{[N]}) - \mathcal{L}_{\mu_{[N]}}(\theta^*) = \frac{\|y - X\bar{\theta}_{[N]}\|_2}{\sqrt{n}} - \frac{\|\epsilon\|_2}{\sqrt{n}}. \quad (\text{D.4})$$

From Wainwright (2015), we have for any  $\delta > 0$ ,

$$\mathbb{P}\left[\left|\frac{\|\epsilon\|_2^2}{n} - \sigma^2\right| \geq \sigma^2 \delta\right] \leq 2 \exp\left(-\frac{n\delta^2}{18}\right). \quad (\text{D.5})$$

Combining (D.2), (D.3), (D.4) and (D.5) with  $\delta^2 = \frac{c_3 s^* \log d}{n}$ , we have with high probability,

$$\left|\frac{\|y - X\bar{\theta}_{[N]}\|_2}{\sqrt{n}} - \sigma\right| = \mathcal{O}\left(\frac{\sigma s^* \log d}{n}\right). \quad (\text{D.6})$$

From Part 1, for  $n \geq c_4 s^* \log d$ , we have with high probability,

$$\|\bar{\theta}_{[N]} - \theta^*\|_2 \leq \frac{384\sigma \sqrt{s^* \log d/n}}{\psi_{\min}} \leq \frac{\sigma}{2\sqrt{2}\psi_{\max}},$$

then  $\bar{\theta}_{[N]} \in \mathcal{B}_r^{s^* + \bar{s}}$  and  $\|\bar{\theta}_{[N]} - \bar{\theta}_{[N]}\|_0 \leq s^* + 2\bar{s}$ . Then from the analysis of Theorem B.2, we have

$$\omega_{\mu_{[N]}, \lambda_{[N]}}(\theta_{[N]}^{(t+1)}) \leq (1 + \kappa_{s^* + 2\bar{s}}) \sqrt{4\rho_{s^* + 2\bar{s}}^+ \left( \mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\theta_{[N]}^{(t)}) - \mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\bar{\theta}_{[N]}) \right)} \leq \varepsilon_{[N]}.$$

This implies

$$\mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\theta_{[N]}^{(t)}) - \mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\bar{\theta}_{[N]}) \leq \frac{\varepsilon_{[N]}^2}{4\rho_{s^*+2\tilde{s}}^+(1 + \kappa_{s^*+2\tilde{s}})^2}. \quad (\text{D.7})$$

On the other hand, from the LRSC property of  $\mathcal{L}_{\mu_{[N]}}$ , convexity of  $\ell_1$  norm and optimality of  $\bar{\theta}$ , we have

$$\mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\theta_{[N]}^{(t)}) - \mathcal{F}_{\mu_{[N]}, \lambda_{[N]}}(\bar{\theta}_{[N]}) \geq \rho_{s^*+2\tilde{s}}^- \|\widehat{\theta}_{[N]} - \bar{\theta}_{[N]}\|_2^2. \quad (\text{D.8})$$

Combining (D.7), (D.8) and Assumption 3.5, we have

$$\begin{aligned} \frac{\|X(\widehat{\theta}_{[N]} - \bar{\theta}_{[N]})\|_2}{\sqrt{n}} &\leq \sqrt{\frac{8\rho_{s^*+2\tilde{s}}^+}{\sigma}} \|\widehat{\theta}_{[N]} - \theta^*\|_2 \leq \sqrt{\frac{2}{\sigma\rho_{s^*+2\tilde{s}}^-}} \frac{\varepsilon_{[N]}}{(1 + \kappa_{s^*+2\tilde{s}})} \\ &\leq \frac{4\varepsilon_{[N]}}{(1 + \kappa_{s^*+2\tilde{s}})\sqrt{\psi_{\min}}}. \end{aligned} \quad (\text{D.9})$$

Combining (D.6) and (D.9), we have

$$\begin{aligned} \left| \frac{\|y - X\widehat{\theta}_{[N]}\|_2}{\sqrt{n}} \right| &\leq \left| \frac{\|y - X\bar{\theta}_{[N]}\|_2}{\sqrt{n}} \right| + \frac{\|X(\widehat{\theta}_{[N]} - \bar{\theta}_{[N]})\|_2}{\sqrt{n}} \\ &\leq \left| \frac{\|y - X\bar{\theta}_{[N]}\|_2}{\sqrt{n}} \right| + \frac{4\varepsilon_{[N]}}{(1 + \kappa_{s^*+2\tilde{s}})\sqrt{\psi_{\min}}}. \end{aligned}$$

If  $\varepsilon_{[N]} \leq c_5 \frac{\sigma s^* \log d}{n}$  for some constant  $c_5$ , then we have the desired result.

## E Proof of Proposition 3.8

Let  $\widetilde{\theta}_{[K]}$  and  $\bar{\theta}_{[K]}$  be the unique global optima of (1.3) and (1.4) respectively for all  $K = N_1 + 1, \dots, N$ . Then, we show  $\widetilde{\theta}_{[K]} = \bar{\theta}_{[K]}$  under the proposed conditions. Apply the argument of the proof of Theorem 3.7 recursively, we have with probability at least  $1 - 8\exp\left(-\frac{n}{144}\right) - 2d^{-1}$ ,

$$\|\bar{\theta}_{[K]} - \theta^*\|_2 \leq \frac{384\sigma\sqrt{s^*\log d/n}}{\psi_{\min}}.$$

By SE condition of  $X$  in Assumption 3.3, this implies

$$\|X(\bar{\theta}_{[K]} - \theta^*)\|_2 \leq \frac{384\sigma\sqrt{2\psi_{\max}s^*\log d}}{\psi_{\min}}. \quad (\text{E.1})$$

On the other hand, we have

$$\|y - X\bar{\theta}_{[K]}\|_2 = \|X(\bar{\theta}_{[K]} - \theta^*) + \epsilon\|_2 \geq \|\epsilon\|_2 - \|X(\bar{\theta}_{[K]} - \theta^*)\|_2. \quad (\text{E.2})$$

Since  $\epsilon$  has i.i.d. sub-Gaussian entries with  $\mathbb{E}[\epsilon_i] = 0$  and  $\mathbb{E}[\epsilon_i^2] = \sigma^2$  for all  $i = 1, \dots, n$ , we have from [Wainwright \(2015\)](#) that

$$\mathbb{P}\left[\|\epsilon\|_2^2 \leq \frac{2}{3}n\sigma^2\right] \leq \exp\left(-\frac{n}{144}\right), \quad (\text{E.3})$$

Note that we choose  $\mu_{[N]}$  sufficiently small such that  $\mu_{[K]} \leq \sqrt{n}\sigma/4$  for all  $K \geq N_1$ . Combining (E.1) and (E.2), (E.3) and the condition on  $n \geq c_4 s^* \log d$  for some constant  $c_4$ , we have with probability at least  $1 - 9\exp\left(-\frac{n}{144}\right) - 2d^{-1}$ ,

$$\|y - X\bar{\theta}_{[K]}\|_2 \geq \sqrt{n}\sigma \left( \sqrt{\frac{2}{3}} - \frac{384\sigma \sqrt{2\psi_{\max}s^* \log d/n}}{\psi_{\min}} \right) > \frac{\sqrt{n}\sigma}{4} \geq \mu_{[K]}.$$

This implies  $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta) = \mathcal{F}_{\mu_{[K]}}(\theta) + \frac{\mu_{[K]}}{2}$ , thus  $\arg\min_{\theta} \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta) = \arg\min_{\theta} \mathcal{F}_{\mu_{[K]}}(\theta)$ , i.e.,  $\bar{\theta}_{[K]} = \bar{\theta}_{[K]}$ . In addition this also implies  $\|y - X\theta^*\|_2 > \frac{\sqrt{n}\sigma}{4} \geq \mu_{[K]}$ , i.e.,  $\theta^*$  is not in the smoothed region.

Applying the same argument again to  $\hat{\theta}_{[K]}$ , we have that for large enough  $n$ , with high probability,

$$\|y - X\hat{\theta}_{[K]}\|_2 > \frac{\sqrt{n}\sigma}{4} \geq \mu_{[K]},$$

and  $r = \frac{\sigma^2}{8\psi_{\max}} > s^* \left(64\sigma \lambda_{[N_1]}/\psi_{\min}\right)^2 \geq s^* \left(8\lambda_{[N_1]}/\rho_{s^*+\bar{s}}^-\right)^2$  is guaranteed, where  $\lambda_{[N_1]} \geq 2\lambda_{[N]} = 48\sqrt{\log d/n}$ . By Lemma B.4, this implies the existence of the linear convergence region, which does not fall into the smoothed region. In addition,  $\hat{\theta}_{[K]}$  is not in the smoothed region.

## F Proof of Theorem B.1

The sublinear rate of convergence (B.1) follows directly from Lemma A.7. In terms of the approximate KKT condition, we have

$$\begin{aligned} \omega_{\mu_{[K]}, \lambda_{[K]}}^2(\theta_{[K]}^{(t+1)}) &\stackrel{(i)}{\leq} \left( \tilde{L}_{[K]}^{(t)} + H_{\tilde{L}_{[K]}^{(t)}}(\theta_{[K]}^{(t)}) \right)^2 \|\theta_{[K]}^{(t+1)} - \theta_{[K]}^{(t)}\|_2^2 \\ &\stackrel{(ii)}{\leq} \left( \tilde{L}_{[K]}^{(t)} + \|X\|_2^2/(\sqrt{n}\mu_{[K]}) \right)^2 \|\theta_{[K]}^{(t+1)} - \theta_{[K]}^{(t)}\|_2^2 \\ &\stackrel{(iii)}{\leq} \frac{2 \left( \tilde{L}_{[K]}^{(t)} + \|X\|_2^2/(\sqrt{n}\mu_{[K]}) \right)^2}{(t-m+1)} \cdot \frac{\left( \sum_{i=m}^t \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(i)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(i+1)}) \right)}{\min_{i \in \{m, \dots, t\}} \{\tilde{L}_{[K]}^{(i)}\}} \\ &\stackrel{(iv)}{\leq} \frac{18\|X\|_2^4}{(t-m+1)n\mu_{[K]}^2} \cdot \frac{\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(m)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t+1)})}{L_{\mu_{[K]}}} \\ &\leq \frac{18\|X\|_2^4}{(t-m+1)n\mu_{[K]}^2} \cdot \frac{\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(m)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]})}{L_{\mu_{[K]}}} \\ &\stackrel{(v)}{\leq} \frac{72R^2\|X\|_2^6}{L_{\mu_{[K]}}(t-m+1)(m+2)n^{3/2}\mu_{[K]}^3} \stackrel{(vi)}{\leq} \frac{288R^2\|X\|_2^6}{L_{\mu_{[K]}}(t+3)^2n^{3/2}\mu_{[K]}^3}, \end{aligned} \quad (\text{F.1})$$



where (i) is from Lemma A.9, (ii) is from  $H_{\widetilde{L}_{[K]}^{(t)}}(\theta_{[K]}^{(t)}) \leq L_{\mu_{[K]}} \leq \|X\|_2^2/(\sqrt{n}\mu_{[K]})$  by Lemma A.9 and (N.41) in Lemma A.7, (iii) is from (A.16) in Lemma A.8, (iv) is from  $L_{\mu_{[K]}} \leq \widetilde{L}_{[K]}^{(t)} \leq 2L_{\mu_{[K]}} \leq 2\|X\|_2^2/(\sqrt{n}\mu_{[K]})$  in Remark A.1 and (N.41) in Lemma A.7, (v) is from Lemma A.7 and (vi) is obtained by choosing  $m = \lfloor t/2 \rfloor$ . To achieve the approximate KKT condition  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) \leq \varepsilon_{[K]}$ , we require the R.H.S. of (F.1) to be no greater than  $\varepsilon_{[K]}^2$ , then we have the desired result (B.2).

## G Proof of Theorem B.2

Note that the RSS property implies that line search terminate when  $\widetilde{L}_{[K]}^{(t)}$  satisfies

$$\rho_{s^*+2\widetilde{s}}^+ \leq \widetilde{L}_{[K]}^{(t)} \leq 2\rho_{s^*+2\widetilde{s}}^+ \quad (\text{G.1})$$

From Lemma B.3, the initialization  $\theta_{[K]}^{(0)}$  satisfies  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) \leq \lambda_{[K]}/2$  with  $\|(\theta_{[K]}^{(0)})_{\widetilde{S}^*}\|_0 \leq \widetilde{s}$ . Then from Lemma A.2, the objective satisfies

$$\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta^*) \leq \frac{6\lambda_{[K]}^2 s^*}{\rho_{s^*+2\widetilde{s}}^-}.$$

Then by Lemma A.4, we have

$$\|(\theta_{[K]}^{(1)})_{\widetilde{S}^*}\|_0 \leq \widetilde{s}.$$

By the monotone decrease of  $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)})$  from (A.16) in Lemma A.8 and recursively applying Lemma A.4,  $\|(\theta_{[K]}^{(t)})_{\widetilde{S}^*}\|_0 \leq \widetilde{s}$  holds in (B.3) for any  $t = 1, 2, \dots$

For the objective error, we have

$$\begin{aligned} \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) &\stackrel{(i)}{\leq} \left(1 - \frac{1}{8\kappa_{s^*+2\widetilde{s}}}\right)^t \left(\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(0)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]})\right) \\ &\stackrel{(ii)}{\leq} \left(1 - \frac{1}{8\kappa_{s^*+2\widetilde{s}}}\right)^t \frac{24\lambda_{[K]}s^*\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)})}{\rho_{s^*+2\widetilde{s}}^-}, \end{aligned} \quad (\text{G.2})$$

where (i) is from Lemma A.6, and (ii) is from Lemma A.5 with  $\widetilde{\lambda} = \lambda = \lambda_{[K]}$  and  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t+1)}) \leq \lambda_{[K]}/2 \leq \lambda_{[K]}$ , which results in (B.3).

Combining (G.2) and (A.4), we have

$$\begin{aligned} \|\theta_{[K]}^{(t)} - \bar{\theta}_{[K]}\|_2^2 &\leq \frac{1}{\rho_{s^*+2\widetilde{s}}^-} \left(\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) - \nabla \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]})\right) \\ &= \frac{1}{\rho_{s^*+2\widetilde{s}}^-} \left(\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]})\right) \leq \left(1 - \frac{1}{8\kappa_{s^*+2\widetilde{s}}}\right)^t \frac{24\lambda_{[K]}s^*\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)})}{(\rho_{s^*+2\widetilde{s}}^-)^2} \end{aligned}$$

For the optimal residue  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t+1)})$  of  $(t+1)$ -th iteration of  $K$ -th path following stage, we have

$$\begin{aligned}
\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t+1)}) &\stackrel{(i)}{\leq} \left( \widetilde{L}_{[K]}^{(t)} + H_{\widetilde{L}_{[K]}^{(t)}}(\theta_{[K]}^{(t)}) \right) \|\theta_{[K]}^{(t+1)} - \theta_{[K]}^{(t)}\|_2 \stackrel{(ii)}{\leq} \left( \widetilde{L}_{[K]}^{(t)} + 2\rho_{s^*+2\widetilde{s}}^+ \right) \|\theta_{[K]}^{(t+1)} - \theta_{[K]}^{(t)}\|_2 \\
&\stackrel{(iii)}{\leq} \widetilde{L}_{[K]}^{(t)} \left( 1 + \frac{2\rho_{s^*+2\widetilde{s}}^+}{\rho_{s^*+2\widetilde{s}}^-} \right) \|\theta_{[K]}^{(t+1)} - \theta_{[K]}^{(t)}\|_2 \\
&\stackrel{(iv)}{\leq} \widetilde{L}_{[K]}^{(t)} \left( 1 + \frac{2\rho_{s^*+2\widetilde{s}}^+}{\rho_{s^*+2\widetilde{s}}^-} \right) \sqrt{\frac{2 \left( \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t+1)}) \right)}{\widetilde{L}_{[K]}^{(t)}}} \\
&\stackrel{(v)}{\leq} (1 + 2\kappa_{s^*+2\widetilde{s}}) \sqrt{4\rho_{s^*+2\widetilde{s}}^+ \left( \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)}) - \mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\bar{\theta}_{[K]}) \right)} \\
&\stackrel{(vi)}{\leq} (1 + 2\kappa_{s^*+2\widetilde{s}}) \sqrt{96\lambda_{[K]}^2 s^* \kappa_{s^*+2\widetilde{s}} \left( 1 - \frac{1}{8\kappa_{s^*+2\widetilde{s}}} \right)^t}, \tag{G.3}
\end{aligned}$$

where (i) is from Lemma A.9, (ii) is from  $H_{\widetilde{L}_{[K]}^{(t)}}(\theta_{[K]}^{(t)}) \leq L_{\mu_{[K]}} \leq \rho_{s^*+2\widetilde{s}}^+$  by Lemma A.9 and Remark A.1, (iii) is from  $\rho_{s^*+2\widetilde{s}}^- \leq \widetilde{L}_{[K]}^{(t)}$  in (G.1), (iv) is from (A.16) in Lemma A.8, (v) is from  $\widetilde{L}_{[K]}^{(t)} \leq 2\rho_{s^*+2\widetilde{s}}^+$  in (G.1) and monotone decrease of  $\mathcal{F}_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t)})$  from (A.16) in Lemma A.8, and (vi) is from (G.2) and  $\kappa_{s^*+2\widetilde{s}} = \frac{\rho_{s^*+2\widetilde{s}}^+}{\rho_{s^*+2\widetilde{s}}^-}$ .

For the  $K$ -th path following stage,  $K = 1, \dots, N-1$ , to have  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\theta_{[K]}^{(t+1)}) \leq \lambda_{[K]}/4$ , we set the R.H.S. of (G.3) to be no greater than  $\lambda_{[K]}/4$ , which is equivalent to require the number of iterations in the  $K$ -th stage to be upper bounded by (B.4). For the last  $N$ -th path following stage, we need  $\omega_{\mu_{[N]}, \lambda_{[N]}}(\bar{\theta}_{[N]}) \leq \varepsilon_{[N]} \leq \lambda_{[N]}/4$ . Set the R.H.S. of (G.3) to be no greater than  $\varepsilon_{[N]}$ , which is equivalent to require the number of iterations in the  $K$ -th stage to be an upper bounded by (B.5).

## H Proof of Lemma B.3

Since  $\omega_{\mu_{[K-1]}, \lambda_{[K-1]}}(\widehat{\theta}_{[K-1]}) \leq \lambda_{[K-1]}/4$ , there exists some subgradient  $g \in \partial \|\widehat{\theta}_{[K-1]}\|_1$  such that

$$\|\nabla \mathcal{L}_{\mu_{[K-1]}}(\widehat{\theta}_{[K-1]}) + \lambda_{[K-1]}g\|_\infty \leq \lambda_{[K-1]}/4. \tag{H.1}$$

By the definition of  $\omega_{\mu_{[K]}, \lambda_{[K]}}(\cdot)$ , we have

$$\begin{aligned}
\omega_{\mu_{[K]}, \lambda_{[K]}}(\widehat{\theta}_{[K-1]}) &\leq \|\nabla \mathcal{L}_{\mu_{[K-1]}}(\widehat{\theta}_{[K-1]}) + \lambda_{[K]}g\|_\infty = \|\nabla \mathcal{L}_{\mu_{[K-1]}}(\widehat{\theta}_{[K-1]}) + \lambda_{[K-1]}g + (\lambda_{[K]} - \lambda_{[K-1]})g\|_\infty \\
&\leq \|\nabla \mathcal{L}_{\mu_{[K-1]}}(\widehat{\theta}_{[K-1]}) + \lambda_{[K-1]}g\|_\infty + |\lambda_{[K]} - \lambda_{[K-1]}| \cdot \|g\|_\infty \stackrel{(i)}{\leq} \lambda_{[K-1]}/4 + (1 - \eta_\lambda)\lambda_{[K-1]} \\
&\stackrel{(ii)}{\leq} \lambda_{[K]}/2,
\end{aligned}$$

where (i) is from (H.1) and choice of  $\lambda_{[K]}$ , (ii) is from the condition on  $\eta_\lambda$ .

## I Proof of Lemma B.4

**Part 1.** We first show  $\|\theta - \theta^*\|_2^2 \leq r$  by contradiction. Suppose  $\|\theta - \theta^*\|_2 > \sqrt{r}$ . Let  $\alpha \in [0, 1]$  such that  $\tilde{\theta} = (1 - \alpha)\theta + \alpha\theta^*$  and

$$\|\tilde{\theta} - \theta^*\|_2 = \sqrt{r}. \quad (\text{I.1})$$

Let  $\tilde{g} = \operatorname{argmin}_{g \in \partial\|\theta\|_1} \|\nabla \mathcal{L}_\mu(\theta) + \lambda g\|_\infty$  and  $\Delta = \theta - \theta^*$ , then we have

$$\begin{aligned} \mathcal{F}_{\mu,\lambda}(\theta^*) &\stackrel{(i)}{\geq} \mathcal{F}_{\mu,\lambda}(\theta) - (\nabla \mathcal{L}_\mu(\theta) + \lambda \tilde{g})^\top \Delta \geq \mathcal{F}_{\mu,\lambda}(\theta) - \|\nabla \mathcal{L}_\mu(\theta) + \lambda \tilde{g}\|_\infty \|\Delta\|_1 \\ &\stackrel{(ii)}{\geq} \mathcal{F}_{\mu,\lambda}(\theta) - \frac{\lambda}{4} \|\Delta\|_1, \end{aligned} \quad (\text{I.2})$$

where (i) is from the convexity of  $\mathcal{F}_{\mu,\lambda}(\theta)$  and (ii) is from the approximate KKT condition.

Denote  $\tilde{\Delta} = \tilde{\theta} - \theta^*$ . Combining (I.2) and (I.1), we have

$$\begin{aligned} \mathcal{F}_{\mu,\lambda}(\tilde{\theta}) &\stackrel{(i)}{\leq} (1 - \alpha)\mathcal{F}_{\mu,\lambda}(\theta) + \alpha\mathcal{F}_{\mu,\lambda}(\theta^*) \leq (1 - \alpha)\mathcal{F}_{\mu,\lambda}(\theta^*) + \frac{(1 - \alpha)\lambda}{4} \|\Delta\|_1 + \alpha\mathcal{F}_{\mu,\lambda}(\theta^*) \\ &\leq \mathcal{F}_{\mu,\lambda}(\theta^*) + \frac{\lambda}{4} \|(1 - \alpha)(\theta - \theta^*)\|_1 = \mathcal{F}_{\mu,\lambda}(\theta^*) + \frac{\lambda}{4} \|(1 - \alpha)\theta + \alpha\theta^* - \theta^*\|_1 \\ &= \mathcal{F}_{\mu,\lambda}(\theta^*) + \frac{\lambda}{4} \|\tilde{\theta} - \theta^*\|_1 = \mathcal{F}_{\mu,\lambda}(\theta^*) + \frac{\lambda}{4} \|\tilde{\Delta}\|_1. \end{aligned}$$

where (i) is from the convexity of  $\mathcal{F}_{\mu,\lambda}(\theta)$ . This indicates

$$\begin{aligned} \mathcal{L}_\mu(\tilde{\theta}) - \mathcal{L}_\mu(\theta^*) &\leq \lambda(\|\theta^*\|_1 - \|\tilde{\theta}\|_1) + \frac{1}{4} \|\tilde{\Delta}\|_1 \\ &= \lambda(\|\theta_{\mathcal{S}^*}^*\|_1 - \|\tilde{\theta}_{\mathcal{S}^*}\|_1 - \|\tilde{\theta}_{\mathcal{S}^c}\|_1) + \frac{1}{4} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 + \frac{1}{4} \|\tilde{\Delta}_{\mathcal{S}^c}\|_1 \\ &\leq \lambda(\|\theta_{\mathcal{S}^*}^* - \tilde{\theta}_{\mathcal{S}^*}\|_1 - \|\tilde{\theta}_{\mathcal{S}^c} - \theta_{\mathcal{S}^c}^*\|_1) + \frac{1}{4} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 + \frac{1}{4} \|\tilde{\Delta}_{\mathcal{S}^c}\|_1 \\ &= \frac{5\lambda}{4} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 - \frac{3\lambda}{4} \|\tilde{\Delta}_{\mathcal{S}^c}\|_1. \end{aligned} \quad (\text{I.3})$$

On the other hand, we have

$$\begin{aligned} \mathcal{L}_\mu(\tilde{\theta}) - \mathcal{L}_\mu(\theta^*) &\stackrel{(i)}{\geq} \nabla \mathcal{L}_\mu(\theta^*)^\top \tilde{\Delta} \geq -\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty \|\tilde{\Delta}\|_1 \stackrel{(ii)}{\geq} -\frac{\lambda}{6} \|\tilde{\Delta}\|_1 \\ &= -\frac{\lambda}{6} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 - \frac{\lambda}{6} \|\tilde{\Delta}_{\mathcal{S}^c}\|_1, \end{aligned} \quad (\text{I.4})$$

where (i) is from the convexity of  $\mathcal{L}_\mu(\theta)$ , (ii) is from Assumption 3.2. Combining (I.3) and (I.4), we have

$$\|\tilde{\Delta}_{\mathcal{S}^c}\|_1 \leq \frac{5}{2} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1. \quad (\text{I.5})$$

Next, we consider the following sequence of sets:

$$\mathcal{S}_0 = \left\{ j \in \bar{\mathcal{S}}^* : \sum_{m \in \bar{\mathcal{S}}^*} \mathbb{1}(\tilde{\theta}_m \geq \tilde{\theta}_j) \leq \bar{s} \right\} \text{ and}$$

$$\mathcal{S}_i = \left\{ j \in \bar{\mathcal{S}}^* \setminus \bigcup_{k < i} \mathcal{S}_k : \sum_{m \in \bar{\mathcal{S}}^* \setminus \bigcup_{k < i} \mathcal{S}_k} \mathbb{1}(\tilde{\theta}_m \geq \tilde{\theta}_j) \leq \bar{s} \right\} \text{ for all } i = 1, 2, \dots$$

We introduce a result from [Bühlmann and van de Geer \(2011\)](#) with its proof provided therein.

**Lemma I.1** (Adapted from Lemma 6.9 in [Bühlmann and van de Geer \(2011\)](#) by setting  $q = 2$ ). Let  $v = [v_1, v_2, \dots]^\top$  with  $v_1 \geq v_2 \geq \dots \geq 0$ . For any  $s \in \{1, 2, \dots\}$ , we have

$$\left( \sum_{j \geq s+1} v_j^2 \right)^{1/2} \leq \sum_{k=1}^{\infty} \left( \sum_{j=ks+1}^{(k+1)s} v_j^2 \right)^{1/2} \leq \frac{\|v\|_1}{\sqrt{s}}.$$

Denote  $\mathcal{A} = \mathcal{S}^* \cup \mathcal{S}_0$ . Then we have

$$\sum_{i \geq 1} \|\tilde{\Delta}_{\mathcal{S}_i}\|_2 \stackrel{(i)}{\leq} \frac{1}{\sqrt{\bar{s}}} \|\tilde{\Delta}_{\bar{\mathcal{S}}^*}\|_1 \stackrel{(ii)}{\leq} \frac{5}{2} \sqrt{\frac{s^*}{\bar{s}}} \|\tilde{\Delta}_{\mathcal{S}^*}\|_2 \leq \frac{5}{2} \sqrt{\frac{s^*}{\bar{s}}} \|\tilde{\Delta}_{\mathcal{A}}\|_2, \quad (\text{I.6})$$

where (i) is from Lemma I.1 with  $s = \bar{s}$  and (ii) is from (I.5). Let  $\check{\theta} = (1 - \beta)\tilde{\theta} + \beta\theta^*$  for any  $\beta \in [0, 1]$ . Then we have

$$\|\check{\theta} - \theta^*\|_2 = (1 - \beta)\|\tilde{\theta} - \theta^*\|_2 \leq \sqrt{r},$$

which implies  $\mathcal{L}_\mu(\check{\theta})$  satisfies RSC/RSS for  $\check{\theta}$  restricted on a sparse set by Assumption 3.3. Then we have

$$\begin{aligned} |\tilde{\Delta}_{\mathcal{A}}^\top \nabla_{\bar{\mathcal{A}}, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}}| &\leq \sum_{i \geq 1} |\tilde{\Delta}_{\mathcal{S}_i}^\top \nabla_{\mathcal{S}_i, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}}| \leq \rho_{s^* + \bar{s}}^+ \|\tilde{\Delta}_{\mathcal{A}}\|_2 \sum_{i \geq 1} \|\tilde{\Delta}_{\mathcal{S}_i}\|_2 \\ &\stackrel{(i)}{\leq} \frac{5}{2} \sqrt{\frac{s^*}{\bar{s}}} \rho_{s^* + \bar{s}}^+ \|\tilde{\Delta}_{\mathcal{A}}\|_2^2, \end{aligned} \quad (\text{I.7})$$

where (i) is from (I.6). On the other hand, we have from RSC

$$\tilde{\Delta}_{\mathcal{A}}^\top \nabla_{\mathcal{A}, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}} \geq \rho_{s^* + \bar{s}}^- \|\tilde{\Delta}_{\mathcal{A}}\|_2^2. \quad (\text{I.8})$$

Then we have w.h.p.

$$\begin{aligned} \tilde{\Delta}^\top \nabla \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta} &= \tilde{\Delta}_{\mathcal{A}}^\top \nabla_{\mathcal{A}, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}} + 2\tilde{\Delta}_{\bar{\mathcal{A}}}^\top \nabla_{\bar{\mathcal{A}}, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}} + \tilde{\Delta}_{\bar{\mathcal{A}}}^\top \nabla_{\bar{\mathcal{A}}, \bar{\mathcal{A}}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\bar{\mathcal{A}}} \\ &\geq \tilde{\Delta}_{\mathcal{A}}^\top \nabla_{\mathcal{A}, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}} - 2|\tilde{\Delta}_{\bar{\mathcal{A}}}^\top \nabla_{\bar{\mathcal{A}}, \mathcal{A}} \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta}_{\mathcal{A}}| \\ &\stackrel{(i)}{\geq} \left( \rho_{s^* + \bar{s}}^- - 5\sqrt{\frac{s^*}{\bar{s}}} \rho_{s^* + \bar{s}}^+ \right) \|\tilde{\Delta}_{\mathcal{A}}\|_2^2 \stackrel{(ii)}{\geq} \frac{9}{14} \rho_{s^* + \bar{s}}^- \|\tilde{\Delta}_{\mathcal{A}}\|_2^2, \end{aligned}$$

where (i) is from (I.7) and (I.8), (ii) is from Assumption 3.3. This implies

$$\begin{aligned}\mathcal{L}_\mu(\tilde{\theta}) - \mathcal{L}_\mu(\theta^*) &= \nabla \mathcal{L}_\mu(\theta^*)^\top \tilde{\Delta} + \frac{1}{2} \tilde{\Delta}^\top \nabla^2 \mathcal{L}_\mu(\check{\theta}) \tilde{\Delta} \geq \nabla \mathcal{L}_\mu(\theta^*)^\top \tilde{\Delta} + \frac{9}{28} \rho_{s^*+\tilde{s}}^- \|\tilde{\Delta}_\mathcal{A}\|_2^2 \\ &\stackrel{(i)}{\geq} \frac{9}{28} \rho_{s^*+\tilde{s}}^- \|\tilde{\Delta}_\mathcal{A}\|_2^2 - \frac{\lambda}{6} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 - \frac{\lambda}{6} \|\tilde{\Delta}_{\tilde{\mathcal{S}}^*}\|_1,\end{aligned}\tag{I.9}$$

where (i) is from Assumption 3.2. Combining (I.3) and (I.9), we have

$$\rho_{s^*+\tilde{s}}^- \|\tilde{\Delta}_{\mathcal{S}^*}\|_2^2 \leq \rho_{s^*+\tilde{s}}^- \|\tilde{\Delta}_\mathcal{A}\|_2^2 \leq \frac{8}{3} \lambda \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 \leq \frac{8}{3} \lambda \sqrt{s^*} \|\tilde{\Delta}_{\mathcal{S}^*}\|_2 \leq \frac{8}{3} \lambda \sqrt{s^*} \|\tilde{\Delta}_\mathcal{A}\|_2.$$

This implies

$$\|\tilde{\Delta}_{\mathcal{S}^*}\|_2 \leq \|\tilde{\Delta}_\mathcal{A}\|_2 \leq \frac{8\lambda\sqrt{s^*}}{3\rho_{s^*+\tilde{s}}^-} \text{ and } \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 \leq \frac{8\lambda s^*}{3\rho_{s^*+\tilde{s}}^-}.\tag{I.10}$$

Then we have

$$\|\tilde{\Delta}_{\tilde{\mathcal{A}}}\|_2 \leq \sum_{i \geq 1} \|\tilde{\Delta}_{\mathcal{S}_i}\|_2 \stackrel{(i)}{\leq} \frac{1}{\sqrt{s}} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 \stackrel{(ii)}{\leq} \frac{5}{2} \sqrt{\frac{1}{s^*}} \|\tilde{\Delta}_{\mathcal{S}^*}\|_1 \stackrel{(iii)}{\leq} \frac{20\lambda\sqrt{s^*}}{3\rho_{s^*+\tilde{s}}^-},\tag{I.11}$$

where (i) is from Lemma I.1 with  $s = \tilde{s}$ , (ii) is from (I.5) and  $\tilde{s} \geq s^*$  and (iii) is from (I.10). Combining (I.10) and (I.11), we have

$$\|\tilde{\Delta}\|_2 = \sqrt{\|\tilde{\Delta}_\mathcal{A}\|_2^2 + \|\tilde{\Delta}_{\tilde{\mathcal{A}}}\|_2^2} \leq \frac{8\lambda\sqrt{s^*}}{\rho_{s^*+\tilde{s}}^-} < \sqrt{r},$$

where the last inequality is from the condition  $\frac{\rho_{s^*+\tilde{s}}^-}{8} \sqrt{\frac{r}{s^*}} > \lambda$ . This conflicts with (I.1), which indicates that  $\|\theta - \theta^*\|_2 \leq \sqrt{r}$ .

**Part 2.** We next demonstrate the sparsity of  $\theta$ . From  $\lambda > \lambda_{[N]} \geq 6\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty$ , then we have

$$\left| \left\{ i \in \tilde{\mathcal{S}}^* : |\nabla_i \mathcal{L}_\mu(\theta^*)| \geq \frac{\lambda}{6} \right\} \right| = 0.\tag{I.12}$$

Denote  $\check{\mathcal{S}}_1 = \left\{ i \in \tilde{\mathcal{S}}^* : |\nabla_i \mathcal{L}_\mu(\theta) - \nabla_i \mathcal{L}_\mu(\theta^*)| \geq \frac{2\lambda}{3} \right\}$  and  $\check{s}_1 = |\check{\mathcal{S}}_1|$ . Then there exists some  $b \in \mathbb{R}^d$  such that  $\|b\|_\infty = 1$ ,  $\|b\|_0 \leq \check{s}_1$  and  $b^\top (\nabla \mathcal{L}_\mu(\theta) - \nabla \mathcal{L}_\mu(\theta^*)) \geq \frac{2\lambda\check{s}_1}{3}$ . Then by the mean value theorem, we have for some  $\check{\theta} = (1-\alpha)\theta + \alpha\theta^*$  with  $\alpha \in [0, 1]$ ,  $\nabla \mathcal{L}_\mu(\theta) - \nabla \mathcal{L}_\mu(\theta^*) = \nabla^2 \mathcal{L}_\mu(\check{\theta}) \Delta$ , where  $\Delta = \theta - \theta^*$ . Then we have

$$\begin{aligned}\frac{2\lambda\check{s}_1}{3} &\leq b^\top \nabla^2 \mathcal{L}_\mu(\check{\theta}) \Delta \stackrel{(i)}{\leq} \sqrt{b^\top \nabla^2 \mathcal{L}_\mu(\check{\theta}) b} \sqrt{\Delta^\top \nabla^2 \mathcal{L}_\mu(\check{\theta}) \Delta} \\ &\stackrel{(ii)}{\leq} \sqrt{\check{s}_1 \rho_{\check{s}_1}^+} \sqrt{\Delta^\top (\nabla \mathcal{L}_\mu(\theta) - \nabla \mathcal{L}_\mu(\theta^*))},\end{aligned}\tag{I.13}$$

where (i) is from the generalized Cauchy-Schwarz inequality, (ii) is from the definition of RSS and the fact that  $\|b\|_2 \leq \sqrt{s_1} \|b\|_\infty = \sqrt{s_1}$ . Let  $g$  achieve  $\min_{g \in \partial \|\theta\|_1} \mathcal{F}_{\mu, \lambda}(\theta)$ . Further, we have

$$\begin{aligned} \Delta^\top (\nabla \mathcal{L}_\mu(\theta) - \nabla \mathcal{L}_\mu(\theta^*)) &\leq \|\Delta\|_1 \|\nabla \mathcal{L}_\mu(\theta) - \nabla \mathcal{L}_\mu(\theta^*)\|_\infty \\ &\leq \|\Delta\|_1 (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \|\nabla \mathcal{L}_\mu(\theta)\|_\infty) \\ &\leq \|\Delta\|_1 (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \|\nabla \mathcal{L}_\mu(\theta) + \lambda g\|_\infty + \lambda \|g\|_\infty) \\ &\stackrel{(i)}{\leq} \frac{28\lambda s^*}{3\rho_{s^*+\tilde{s}}^-} \left( \frac{\lambda}{6} + \frac{\lambda}{4} + \lambda \right) \leq \frac{14\lambda^2 s^*}{\rho_{s^*+\tilde{s}}^-}, \end{aligned} \quad (\text{I.14})$$

where (i) is from combining (I.5) and (I.10), condition on  $\lambda$ , approximate KKT condition and  $\|g\|_\infty \leq 1$ . Combining (I.13) and (I.14), we have  $\frac{2\sqrt{s_1}}{3} \leq \sqrt{\frac{14\rho_{s_1}^+ s^*}{\rho_{s^*+\tilde{s}}^-}}$ , which further implies

$$\check{s}_1 \leq \frac{32\rho_{s_1}^+ s^*}{\rho_{s^*+\tilde{s}}^-} \leq 32\kappa_{s^*+2\tilde{s}} s^* \leq \tilde{s}. \quad (\text{I.15})$$

For any  $v \in \mathbb{R}^d$  that satisfies  $\|v\|_0 \leq 1$ , we have

$$\check{\mathcal{S}}_2 = \left\{ i \in \bar{\mathcal{S}}^* : \left| \nabla_i \mathcal{L}_\mu(\theta) + \frac{\lambda}{4} v_i \right| \geq \frac{5\lambda}{6} \right\} \subseteq \left\{ i \in \bar{\mathcal{S}}^* : |\nabla_i \mathcal{L}_\mu(\theta^*)| \geq \frac{\lambda}{6} \right\} \bigcup \check{\mathcal{S}}_1.$$

Then we have  $|\check{\mathcal{S}}_2| \leq |\check{\mathcal{S}}_1| \leq \tilde{s}$ . Since for any  $i \in \bar{\mathcal{S}}^*$  and  $\left| \nabla_i \mathcal{L}_\mu(\theta) + \frac{\lambda}{4} v_i \right| < \frac{5\lambda}{6}$ , we can find  $g_i$  that satisfies  $|g_i| \leq 1$  such that  $\nabla_i \mathcal{L}_\mu(\theta) + \frac{\lambda}{4} v_i + \lambda g_i = 0$  which implies  $\theta_i = 0$ , then we have

$$\left| \left\{ i \in \bar{\mathcal{S}}^* : \left| \nabla_i \mathcal{L}_\mu(\theta) + \frac{\lambda}{4} v_i \right| < \frac{5\lambda}{6} \right\} \right| = 0.$$

Therefore, we have  $\|\theta_{\bar{\mathcal{S}}^*}\|_0 \leq |\check{\mathcal{S}}_2| \leq \tilde{s}$ .

## J Proof of Lemma B.5

Let  $n^{(t)}$  be the number of proximal-gradient steps in  $t$ -th iteration of PIS<sup>2</sup>TA. Then we have

$$L^{(t+1)} \leq 2L^{(t)} \left( \frac{1}{2} \right)^{n^{(t)}-1}.$$

This indicates

$$n^{(t)} \leq 2 + \log_2 \frac{L^{(t)}}{L^{(t+1)}}.$$

Then we have

$$\sum_{t=0}^{T_K} n^{(t)} \leq 2(T_K + 1) + \log_2 \frac{L^{(0)}}{L^{(T_K+1)}}$$

We obtain the desired result by  $L^{(0)} = L_{\max}$  and  $L^{(T_K+1)} \geq \rho_{s^*+2\tilde{s}}^+$ .

## K Intermediate Results of Theorem 4.3

We start with some preliminaries. For any  $\mathcal{S} \subset \{1, \dots, d\}$  with  $|\mathcal{S}| \leq s^*$ , we denote the set of cone

$$\mathcal{C}_{\mathcal{S}}^{\nu} = \left\{x \in \mathbb{R}^d : \|x_{\mathcal{S}^c}\|_1 \leq \nu \|x_{\mathcal{S}}\|_1\right\} \quad \text{and} \quad \mathcal{C}_{s^*}^{\nu} = \bigcup_{\mathcal{S} \subset \{1, \dots, d\}, |\mathcal{S}| \leq s^*} \mathcal{C}_{\mathcal{S}}^{\nu}.$$

In addition, since  $\Theta^* = \Sigma^{*-1} \in \mathcal{M}(\kappa, s^*)$ , we have

$$\frac{\Lambda_{\max}(\Theta^*)}{\Lambda_{\min}(\Theta^*)} = \frac{\Lambda_{\max}(\Sigma^{*-1})}{\Lambda_{\min}(\Sigma^{*-1})} = \frac{\Lambda_{\max}(\Sigma^*)}{\Lambda_{\min}(\Sigma^*)} \leq \kappa$$

We first introduce some important results on characterizing the data matrix  $X$ . These are adapted from intermediate lemmas in [Liu and Wang \(2012\)](#), which we refer to interested readers for detailed proofs.

The first lemma provides the bounds of entry-wise difference between sample and population correlation matrices.

**Lemma K.1.** Let  $\widehat{R}_{\Sigma}$  and  $R_{\Sigma}$  be the sample and population correlation matrices. Then for event

$$\mathcal{E}_1 = \left\{ \|\widehat{R}_{\Sigma} - R_{\Sigma}\|_{\infty} \leq 18 \sqrt{\frac{\log d}{n}} \right\},$$

we have  $\mathbb{P}[\mathcal{E}_1] \geq 1 - d^{-1}$ .

The second lemma provides the bounds of normalized model noise  $\epsilon$ .

**Lemma K.2.** Let  $\epsilon_i \in \mathbb{R}^n$  follows  $\epsilon_i \sim \mathcal{N}_n(0, \sigma_i^2 I_n)$ . Then for event

$$\mathcal{E}_2 = \left\{ \max_{i \in \{1, \dots, d\}} \frac{\|\epsilon_i\|_2^2}{n\sigma_i^2} \leq 1.4 \quad \text{and} \quad \max_{i \in \{1, \dots, d\}} \left\| \frac{\|\epsilon_i\|_2^2}{n\sigma_i^2} - 1 \right\| \leq 3.5 \sqrt{\frac{\log d}{n}} \right\},$$

we have  $\mathbb{P}[\mathcal{E}_2] \geq 1 - d^{-1} - d \exp(-100/n)$ .

The third lemma provides the bounds of sample standard deviation of the marginal univariate Gaussian random variables.

**Lemma K.3.** Let  $\widehat{\Sigma}$  be the sample covariance matrix. Suppose that Assumption 4.1 (A3) holds, then for event

$$\mathcal{E}_3 = \left\{ \frac{1}{2} \Lambda_{\min}(\Sigma) \leq \min_{i \in \{1, \dots, d\}} \widehat{\Sigma}_{ii} \leq \max_{i \in \{1, \dots, d\}} \widehat{\Sigma}_{ii} \leq \frac{3}{2} \Lambda_{\max}(\Sigma) \right\},$$

we have  $\mathbb{P}[\mathcal{E}_3] \geq 1 - d^{-1} - d \exp(-100/n)$ .

Next, we verify Assumption 3.2 in the following lemma, and provide its proof in Appendix N.7.

**Lemma K.4.** Denote  $\mathcal{L}_{\mu,i}(\theta_i^*) = \|z_i - Z_{*\setminus i}\theta_i^*\|_{\mu}/\sqrt{n}$ . Let  $\lambda_{[N]} = \frac{6\sqrt{5\log d/n}}{\max\{\sqrt{5/3}\mu\min_i\{\widehat{\Gamma}_{ii}^{1/2}/(\sqrt{n}\sigma_i)\}, 1\}}$ , then for event

$$\mathcal{E}_4 = \left\{ \lambda_{[N]} \geq 6 \max_{i \in \{1, \dots, d\}} \|\nabla \mathcal{L}_{\mu,i}(\theta_i^*)\|_{\infty} \right\},$$

we have  $\mathbb{P}[\mathcal{E}_4] \geq 1 - d \exp\left(-\frac{n}{25}\right) - \frac{d^{0.6}}{\sqrt{0.6\pi a \log d}}$ .

Then we provide the bound of the restricted eigenvalue of the sample correlation matrix.

**Lemma K.5.** Suppose that  $\mathcal{E}_3$  and Assumption 4.1 (A2) hold, then for event

$$\mathcal{E}_5 = \left\{ \inf_{\theta \in \mathcal{C}_{s^*}^v} \frac{\sqrt{s^* \theta^\top \widehat{R}_{\Sigma} \theta}}{\|R_{\Sigma}\|_1} \geq \frac{1}{5(1+\nu)\sqrt{\kappa}} \right\},$$

there exists constants  $c_1$  and  $c_2$  such that  $\mathbb{P}[\mathcal{E}_5|\mathcal{E}_3] \geq 1 - c_2 \exp(-c_2 n)$ .

Further, we provide the prediction error bound for the approximate solution. It follows directly from Theorem 3.7.

**Lemma K.6.** Suppose that  $\mathcal{E}_5$  holds, then for event

$$\mathcal{E}_6 = \left\{ \max_{i \in \{1, \dots, d\}} \frac{\|Z_{*\setminus i}(\widehat{\theta}_i - \theta_i^*)\|}{\tau_i} \leq c_3 \sqrt{s^* \log d} \right\},$$

there exists constants  $c_1$ ,  $c_2$  and  $c_3$  such that  $\mathbb{P}[\mathcal{E}_6|\mathcal{E}_5] \geq 1 - c_1 \exp(-c_2 n) - c_3 d^{-1}$ .

## L Proof of Lemma 4.2

Using the result in Liu and Wang (2012) (Lemma 12) and Agarwal et al. (2010) (Proposition 1), we have with probability at least  $1 - c_1 \exp(-c_2 n)$  for some universal constants  $c_1$  and  $c_2$ , for any  $v \in \mathbb{R}^d$ ,

$$\frac{1\Lambda_{\min}(\Sigma)}{2\Lambda_{\max}(\Sigma)} \|v\|_2^2 - \frac{9\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)} \cdot \frac{\log d}{n} \|v\|_1^2 \leq \frac{\|Zv\|_2^2}{n} \leq \frac{2\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)} \|v\|_2^2 + \frac{9\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)} \cdot \frac{\log d}{n} \|v\|_1^2.$$

Applying the same analysis for Theorem 3.4, we have for some constant  $c$ ,  $\|v\|_1^2 \leq c\|v_{S^*}\|_1^2 \leq cs^*\|v_{S^*}\|_2^2 \leq cs^*\|v\|_2^2$  and we have

$$\begin{aligned} \left( \frac{1\Lambda_{\min}(\Sigma)}{2\Lambda_{\max}(\Sigma)} - \frac{9\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)} \cdot \frac{s^* \log d}{n} \right) \|v\|_2^2 &\leq \frac{\|Zv\|_2^2}{n} \\ &\leq \left( \frac{2\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)} + \frac{9\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)} \cdot \frac{s^* \log d}{n} \right) \|v\|_2^2. \end{aligned}$$



Since  $n \geq \frac{54\Lambda_{\max}^2(\Sigma)s^*\log d}{\Lambda_{\min}^2(\Sigma)}$ , we have

$$\frac{1\Lambda_{\min}(\Sigma)}{3\Lambda_{\max}(\Sigma)}\|v\|_2^2 \leq \frac{\|Zv\|_2^2}{n} \leq \frac{3\Lambda_{\max}(\Sigma)}{\Lambda_{\min}(\Sigma)}\|v\|_2^2.$$

From  $\Lambda_{\max}(\Sigma) = 1/\Lambda_{\min}(\Theta)$  and  $\Lambda_{\min}(\Sigma) = 1/\Lambda_{\max}(\Theta)$ , we have

$$\frac{1}{3\kappa_{\Theta}}\|v\|_2^2 \leq \frac{\|Zv\|_2^2}{n} \leq 3\kappa_{\Theta}\|v\|_2^2.$$

To satisfy the condition for the computational theory, we require  $\mu_{[K]} \leq \frac{\sqrt{n\text{Var}(\widehat{\Gamma}_{ii}^{-1/2}\epsilon_i)}}{4}$  for all  $i \in \{1, \dots, d\}$  and all  $N_1 \leq K \leq N$ . From  $\sigma_i = \Theta_{ii}^{-1/2}$  and  $\min_{i \in \{1, \dots, d\}} \widehat{\Sigma}_{ii} \leq \frac{3}{2}\Lambda_{\max}(\Sigma)$  with high probability in Lemma K.3, we have

$$\sqrt{\text{Var}(\widehat{\Gamma}_{ii}^{-1/2}\epsilon_i)} = \widehat{\Gamma}_{ii}^{-1/2}\sigma_i = \frac{1}{\sqrt{\Theta_{ii}\widehat{\Sigma}_{ii}}} \geq \frac{1}{\sqrt{\frac{3}{2}\Theta_{ii}\Lambda_{\max}(\Sigma)}}$$

Since  $\Sigma = \Theta^{-1}$ , for any  $i \in \{1, \dots, d\}$ , we choose  $\mu_{[N]}$  sufficiently small such that  $\mu_{[K]}$  satisfies

$$\mu_{[K]} \leq \frac{\sqrt{n}}{4\sqrt{\frac{3}{2}\max_i \Theta_{ii}\Lambda_{\max}(\Sigma)}} \leq \frac{1}{5}\sqrt{\frac{n}{\Lambda_{\max}(\Theta)\Lambda_{\max}(\Sigma)}} = \frac{1}{5}\sqrt{\frac{n\Lambda_{\min}(\Theta)}{\Lambda_{\max}(\Theta)}} = \frac{1}{5}\sqrt{\frac{n}{\kappa_{\Theta}}}.$$

By Lemma K.4, the condition on  $\lambda_{[N]}$  guarantees that Assumption 3.2 holds for each  $i = 1, \dots, d$ . Moreover, when  $n$  is large enough, it can be guaranteed that there exists  $N_1 < N$ ,  $N_1 \in \mathbb{Z}^+$ , such that  $\frac{\rho_{s^*+\bar{s}}}{8}\sqrt{\frac{T}{s^*}} > \lambda_{[N_1]} \geq 2\lambda_{[N]} \geq 12\|\nabla \mathcal{L}_{\mu_{[N]},i}(\theta_i)\|_{\infty}$ .

In addition, following the analysis of Lemma 3.6, we can verify Assumption 3.3 immediately.

## M Proof of Theorem 4.3

The analysis here follows directly from our analysis in the linear model and the analysis in Liu and Wang (2012). Let  $\mathcal{E} = \cap_{i=1}^6 \mathcal{E}_i$ . Combining Lemma K.4 and a sufficiently small value of  $\mu_{[N]}$ , we have  $\lambda_{[N]} = 6\sqrt{\frac{5\log d}{n}}$ . We first show that the estimation error of diagonal elements are bounded.

**Lemma M.1** (Adapted from Lemma 14 in Liu and Wang (2012)). Suppose that Assumption 4.1 and the event  $\mathcal{E}$  hold, then we have

$$\max_{i \in \{1, \dots, d\}} |\widehat{\Theta}_{ii} - \Theta_{ii}^*| \leq c_4 \|\Theta^*\|_2 \frac{\log d}{n}.$$

In addition, we have the  $\ell_1$  norm error bounded for the estimation of off-diagonal elements each column.

**Lemma M.2** (Adapted from Lemma 15 in [Liu and Wang \(2012\)](#)). Suppose that Assumption 4.1 and the event  $\mathcal{E}$  hold, then we have

$$\max_{i \in \{1, \dots, d\}} \|\widehat{\Theta}_{\setminus ii} - \Theta_{\setminus ii}^*\|_1 \leq c_5(s^* \|\Theta^*\|_2 + \|\Theta^*\|_1) \frac{\log d}{n}.$$

Combining Lemma M.1 and Lemma M.2, we have

$$\begin{aligned} \|\widehat{\Theta} - \Theta^*\|_1 &= \max_{i \in \{1, \dots, d\}} \|\widehat{\Theta}_{*i} - \Theta_{*i}^*\|_1 \leq \max_{i \in \{1, \dots, d\}} |\widehat{\Theta}_{ii} - \Theta_{ii}^*| + \|\widehat{\Theta}_{\setminus ii} - \Theta_{\setminus ii}^*\|_1 \\ &\leq c_6(s^* \|\Theta^*\|_2 + \|\Theta^*\|_1) \frac{\log d}{n} \stackrel{(i)}{\leq} c_7(s^* \|\Theta^*\|_2) \frac{\log d}{n}, \end{aligned}$$

where (i) is from  $\|\Theta^*\|_1 \leq s^* \|\Theta^*\|_2$ . Then we finish the proof from

$$\|\widehat{\Theta} - \Theta^*\|_2 \leq \|\widehat{\Theta} - \Theta^*\|_1.$$

## N Proofs of Intermediate Lemmas in Appendix A and Appendix K

### N.1 Proof of Lemma A.2

We first bound the estimation error. From Assumption 3.3, we have the RSC property, which indicates

$$\mathcal{L}_\mu(\theta) \geq \mathcal{L}_\mu(\theta^*) + (\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta^*) + (\rho_{s^*+2\bar{s}}^-/2) \|\theta - \theta^*\|_2^2, \quad (\text{N.1})$$

$$\mathcal{L}_\mu(\theta^*) \geq \mathcal{L}_\mu(\theta) + (\theta^* - \theta)^\top \nabla \mathcal{L}_\mu(\theta) + (\rho_{s^*+2\bar{s}}^-/2) \|\theta - \theta^*\|_2^2, \quad (\text{N.2})$$

Adding (N.2) and (N.1), we have

$$(\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta) \geq (\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta^*) + \rho_{s^*+2\bar{s}}^- \|\theta - \theta^*\|_2^2. \quad (\text{N.3})$$

Let  $g \in \partial \|\theta\|_1$  be the subgradient that achieves the approximate KKT condition of the L.H.S of (A.6), then we have

$$(\theta - \theta^*)^\top (\nabla \mathcal{L}_\mu(\theta) + \lambda g) \leq \|\theta - \theta^*\|_1 \|\nabla \mathcal{L}_\mu(\theta) + \lambda g\|_\infty \leq \frac{1}{2} \lambda \|\theta - \theta^*\|_1. \quad (\text{N.4})$$

On the other hand, we have from (N.3) that

$$(\theta - \theta^*)^\top (\nabla \mathcal{L}_\mu(\theta) + \lambda g) \geq (\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta^*) + \rho_{s^*+2\bar{s}}^- \|\theta - \theta^*\|_2^2 + \lambda g^\top (\theta - \theta^*), \quad (\text{N.5})$$

Since  $\|\theta - \theta^*\|_1 = \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + \|(\theta - \theta^*)_{\bar{\mathcal{S}}^*}\|_1$ , then

$$(\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta^*) \geq -\|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \|\mathcal{L}_\mu(\theta^*)\|_\infty - \|(\theta - \theta^*)_{\bar{\mathcal{S}}^*}\|_1 \|\mathcal{L}_\mu(\theta^*)\|_\infty. \quad (\text{N.6})$$

In addition, we have

$$\begin{aligned} (\theta - \theta^*)^\top g &= g_{\mathcal{S}^*}^\top (\theta - \theta^*)_{\mathcal{S}^*} + g_{\bar{\mathcal{S}}^*}^\top (\theta - \theta^*)_{\bar{\mathcal{S}}^*} \stackrel{(i)}{\geq} -\|g_{\mathcal{S}^*}\|_\infty \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + g_{\bar{\mathcal{S}}^*}^\top \theta_{\bar{\mathcal{S}}^*} \\ &\stackrel{(ii)}{\geq} -\|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + \|g_{\bar{\mathcal{S}}^*}\|_1 \stackrel{(iii)}{=} -\|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + \|(\theta - \theta^*)_{\bar{\mathcal{S}}^*}\|_1, \end{aligned} \quad (\text{N.7})$$

where (i) and (iii) are from  $\theta_{\bar{S}^*}^* = 0$ , (ii) is from  $\|g_{S^*}\|_\infty \leq 1$  and  $g \in \partial\|\theta\|_1$ .

Combining (N.4), (N.5), (N.6), and (N.7), we have

$$\begin{aligned} \frac{1}{2}\lambda\|\theta - \theta^*\|_1 &= \frac{1}{2}\lambda\|(\theta - \theta^*)_{S^*}\|_1 + \frac{1}{2}\lambda\|(\theta - \theta^*)_{\bar{S}^*}\|_1 \\ &\geq \rho_{s^*+2\bar{s}}^-\|\theta - \theta^*\|_2^2 - (\lambda + \|\mathcal{L}_\mu(\theta^*)\|_\infty)\|(\theta - \theta^*)_{S^*}\|_1 \\ &\quad + (\lambda - \|\mathcal{L}_\mu(\theta^*)\|_\infty)\|(\theta - \theta^*)_{\bar{S}^*}\|_1. \end{aligned}$$

This implies

$$\begin{aligned} \rho_{s^*+2\bar{s}}^-\|\theta - \theta^*\|_2^2 + \left(\frac{1}{2}\lambda - \|\mathcal{L}_\mu(\theta^*)\|_\infty\right)\|(\theta - \theta^*)_{\bar{S}^*}\|_1 \\ \leq \left(\frac{3}{2}\lambda + \|\mathcal{L}_\mu(\theta^*)\|_\infty\right)\|(\theta - \theta^*)_{S^*}\|_1, \end{aligned} \quad (\text{N.8})$$

which results in (A.7) from  $\rho_{s^*+2\bar{s}}^- > 0$  and Assumption 3.2 as

$$\|(\theta - \theta^*)_{\bar{S}^*}\|_1 \leq \frac{\frac{3}{2}\lambda + \|\mathcal{L}_\mu(\theta^*)\|_\infty}{\frac{1}{2}\lambda - \|\mathcal{L}_\mu(\theta^*)\|_\infty}\|(\theta - \theta^*)_{S^*}\|_1 \leq 5\|(\theta - \theta^*)_{S^*}\|_1.$$

Combining  $\frac{1}{2}\lambda - \|\mathcal{L}_\mu(\theta^*)\|_\infty \geq 0$ ,  $\frac{3}{2}\lambda + \|\mathcal{L}_\mu(\theta^*)\|_\infty \leq 2\lambda$ , and (N.8), we have estimation error bound in (A.8) and (A.9) as

$$\begin{aligned} \rho_{s^*+2\bar{s}}^-\|\theta - \theta^*\|_2^2 &\leq 2\lambda\|(\theta - \theta^*)_{S^*}\|_1 \leq 2\lambda\sqrt{s^*}\|\theta - \theta^*\|_2. \\ \|\theta - \theta^*\|_1 &\leq 6\|(\theta - \theta^*)_{S^*}\|_1 \leq 6\sqrt{s^*}\|\theta - \theta^*\|_2. \end{aligned}$$

Next, we bound the objective error in (A.10). We have

$$\begin{aligned} \mathcal{F}_{\mu,\lambda}(\theta) - \mathcal{F}_{\mu,\lambda}(\theta^*) &\stackrel{(i)}{\leq} -(\nabla\mathcal{L}_\mu(\theta) + \lambda g)^\top(\theta^* - \theta) \leq \|\nabla\mathcal{L}_\mu(\theta) + \lambda g\|_\infty\|\theta^* - \theta\|_1 \\ &\leq \frac{1}{2}\lambda\|\theta^* - \theta\|_1 = \frac{1}{2}\lambda(\|(\theta^* - \theta)_{S^*}\|_1 + \|(\theta^* - \theta)_{\bar{S}^*}\|_1) \\ &\stackrel{(ii)}{\leq} 3\lambda\|(\theta^* - \theta)_{S^*}\|_1 \leq 3\lambda\sqrt{s^*}\|(\theta^* - \theta)_{S^*}\|_2 \stackrel{(iii)}{\leq} \frac{6\lambda^2 s^*}{\rho_{s^*+2\bar{s}}^-}, \end{aligned}$$

where (i) is from the convexity of  $\mathcal{F}_{\mu,\lambda}(\theta)$  with  $\nabla\mathcal{L}_\mu(\theta) + \lambda g$  as its subgradient, (ii) is from (A.7), and (iii) is from (A.8).

## N.2 Proof of Lemma A.3

Assumption  $\mathcal{F}_{\mu,\lambda}(\theta) - \mathcal{F}_{\mu,\lambda}(\theta^*) \leq 6\lambda^2 s^* / \rho_{s^*+2\bar{s}}^-$  implies

$$\mathcal{L}_\mu(\theta) - \mathcal{L}_\mu(\theta^*) + \lambda(\|\theta\|_1 - \|\theta^*\|_1) \leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\bar{s}}^-}. \quad (\text{N.9})$$

We have from the RSC property that

$$\mathcal{L}_\mu(\theta) \geq \mathcal{L}_\mu(\theta^*) + (\theta - \theta^*)^\top \nabla\mathcal{L}_\mu(\theta^*) + \frac{\rho_{s^*+2\bar{s}}^-}{2}\|\theta - \theta^*\|_2^2, \quad (\text{N.10})$$

Then we have (N.9) and (N.10),

$$\frac{\rho_{s^*+2\tilde{s}}^-}{2} \|\theta - \theta^*\|_2^2 \leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-} - (\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta^*) + \lambda(\|\theta^*\|_1 - \|\theta\|_1). \quad (\text{N.11})$$

In addition, we have

$$(\theta - \theta^*)^\top \nabla \mathcal{L}_\mu(\theta^*) \geq -\|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \|\mathcal{L}_\mu(\theta^*)\|_\infty - \|(\theta - \theta^*)_{\tilde{\mathcal{S}}^*}\|_1 \|\mathcal{L}_\mu(\theta^*)\|_\infty, \quad (\text{N.12})$$

and

$$\|\theta^*\|_1 - \|\theta\|_1 = \|\theta_{\mathcal{S}^*}^*\|_1 - \|\theta_{\mathcal{S}^*}\|_1 - \|(\theta - \theta^*)_{\tilde{\mathcal{S}}^*}\|_1 \leq \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 - \|(\theta - \theta^*)_{\tilde{\mathcal{S}}^*}\|_1. \quad (\text{N.13})$$

Combining (N.11), (N.12) and (N.13), we have

$$\begin{aligned} \frac{\rho_{s^*+2\tilde{s}}^-}{2} \|\theta - \theta^*\|_2^2 &\leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-} + (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \lambda) \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \\ &\quad + (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty - \lambda) \|(\theta - \theta^*)_{\tilde{\mathcal{S}}^*}\|_1. \end{aligned} \quad (\text{N.14})$$

We discuss two cases as following:

**Case 1.** We first assume  $\|\theta - \theta^*\|_1 \leq \frac{12\lambda s^*}{\rho_{s^*+2\tilde{s}}^-}$ . Then (N.14) implies

$$\begin{aligned} \frac{\rho_{s^*+2\tilde{s}}^-}{2} \|\theta - \theta^*\|_2^2 &\stackrel{(i)}{\leq} \frac{6\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-} + (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \lambda) \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \\ &\stackrel{(ii)}{\leq} \frac{6\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-} + \frac{3}{2} \lambda \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \\ &\leq \frac{6\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-} + \frac{18\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-} = \frac{24\lambda^2 s^*}{\rho_{s^*+2\tilde{s}}^-}. \end{aligned}$$

where (i) is from  $\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty - \lambda \leq 0$  and (ii) is from  $\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \lambda \leq \frac{3}{2} \lambda$ . This indicates

$$\|\theta - \theta^*\|_2 \leq \frac{4\sqrt{3s^*}\lambda}{\rho_{s^*+2\tilde{s}}^-}. \quad (\text{N.15})$$

**Case 2.** Next, we assume  $\|\theta - \theta^*\|_1 > \frac{12\lambda s^*}{\rho_{s^*+2\tilde{s}}^-}$ . Then (N.14) implies

$$\begin{aligned} &\frac{\rho_{s^*+2\tilde{s}}^-}{2} \|\theta - \theta^*\|_2^2 \\ &\leq (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \lambda) \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty - \lambda) \|(\theta - \theta^*)_{\tilde{\mathcal{S}}^*}\|_1 + \frac{1}{2} \lambda \|\theta - \theta^*\|_1 \\ &= (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty + \frac{3}{2} \lambda) \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + (\|\nabla \mathcal{L}_\mu(\theta^*)\|_\infty - \frac{1}{2} \lambda) \|(\theta - \theta^*)_{\tilde{\mathcal{S}}^*}\|_1 \\ &\stackrel{(i)}{\leq} 2\lambda \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \leq 2\sqrt{s^*} \lambda \|(\theta - \theta^*)_{\mathcal{S}^*}\|_2, \end{aligned} \quad (\text{N.16})$$

where (i) is from  $\|\nabla\mathcal{L}_\mu(\theta^*)\|_\infty + \frac{3}{2}\lambda \leq 2\lambda$  and  $\|\nabla\mathcal{L}_\mu(\theta^*)\|_\infty - \frac{1}{2}\lambda \leq 0$ . This indicates

$$\|\theta - \theta^*\|_2 \leq \frac{4\sqrt{s^*}\lambda}{\rho_{s^*+2\bar{s}}^-}. \quad (\text{N.17})$$

In addition, we have

$$\|\theta - \theta^*\|_1 \stackrel{(i)}{\leq} 6\|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 \leq 6\sqrt{s^*}\|(\theta - \theta^*)_{\mathcal{S}^*}\|_2 \leq \frac{24\lambda s^*}{\rho_{s^*+2\bar{s}}^-}, \quad (\text{N.18})$$

where (i) is from  $\|\nabla\mathcal{L}_\mu(\theta^*)\|_\infty + \frac{3}{2}\lambda \leq 2\lambda$  and (N.16).

Combining (N.15) and (N.17), we have desired result (A.11). Combining the assumption in Case 1 and (N.18), we have desired result (A.12).

### N.3 Proof of Lemma A.4

Recall that the proximal-gradient update can be computed by the soft-thresholding operation, i.e., for all  $i = 1, \dots, d$ ,

$$\left(\mathcal{T}_{L,\mu,\lambda}(\theta)\right)_i = \text{sign}(\check{\theta}_i) \max\{|\check{\theta}_i| - \lambda/L, 0\} \quad (\text{N.19})$$

where  $\check{\theta} = \theta - \nabla\mathcal{L}_\mu(\theta)/L$ . To bound  $\|(\mathcal{T}_{L,\mu,\lambda}(\theta))_{\bar{\mathcal{S}}^*}\|_0$ , we consider

$$\check{\theta} = \theta - \frac{1}{L}\nabla\mathcal{L}_\mu(\theta) = \theta - \frac{1}{L}\nabla\mathcal{L}_\mu(\theta^*) + \frac{1}{L}(\nabla\mathcal{L}_\mu(\theta^*) - \nabla\mathcal{L}_\mu(\theta)). \quad (\text{N.20})$$

We then consider the following three events:

$$A_1 = \{i \in \bar{\mathcal{S}}^* : |\theta_i| \geq \lambda/(3L)\}, \quad (\text{N.21})$$

$$A_2 = \{i \in \bar{\mathcal{S}}^* : |(\nabla\mathcal{L}_\mu(\theta^*)/L)_i| > \lambda/(6L)\}, \quad (\text{N.22})$$

$$A_3 = \{i \in \bar{\mathcal{S}}^* : |(\nabla\mathcal{L}_\mu(\theta^*)/L - \nabla\mathcal{L}_\mu(\theta)/L)_i| \geq \lambda/(2L)\}, \quad (\text{N.23})$$

**Event  $A_1$ .** Note that for any  $i \in \bar{\mathcal{S}}^*$ ,  $|\theta_i| = |\theta_i - \theta_i^*|$ , then we have

$$\begin{aligned} |A_1| &\leq \sum_{i \in \bar{\mathcal{S}}^*} \frac{3L}{\lambda} |\theta_i - \theta_i^*| \cdot \mathbb{1}(|\theta_i - \theta_i^*| \geq \lambda/(3L)) \leq \frac{3L}{\lambda} \sum_{i \in \bar{\mathcal{S}}^*} |\theta_i - \theta_i^*| \\ &\leq \frac{3L}{\lambda} \|\theta - \theta^*\|_1 \stackrel{(i)}{\leq} \frac{72Ls^*}{\rho_{s^*+2\bar{s}}^-}, \end{aligned} \quad (\text{N.24})$$

where (i) is from (A.12) in Lemma A.3.

**Event  $A_2$ .** By Assumption 3.2 and  $\lambda \geq \lambda_{[N]}$ , we have

$$\begin{aligned} 0 \leq |A_2| &\leq \sum_{i \in \bar{\mathcal{S}}^*} \frac{6L}{\lambda} |(\nabla\mathcal{L}_\mu(\theta^*)/L)_i| \cdot \mathbb{1}(|(\nabla\mathcal{L}_\mu(\theta^*)/L)_i| > \lambda/(6L)) \\ &= \sum_{i \in \bar{\mathcal{S}}^*} \frac{6L}{\lambda} |(\nabla\mathcal{L}_\mu(\theta^*)/L)_i| \cdot 0 = 0, \end{aligned} \quad (\text{N.25})$$

which indicates that  $|A_2| = 0$ .

**Event  $A_3$ .** Consider the event  $\tilde{A} = \left\{ i : \left| \left( \nabla \mathcal{L}_\mu(\theta^*) - \nabla \mathcal{L}_\mu(\theta) \right)_i \right| \geq \lambda/2 \right\}$ , which satisfies  $A_3 \subseteq \tilde{A}$ . We will provide an upper bound of  $|\tilde{A}|$ , which is also an upper bound of  $|A_3|$ . Let  $v \in \mathbb{R}^d$  be chosen such that,  $v_i = \text{sign} \left\{ \left( \nabla \mathcal{L}_\mu(\theta^*)/L - \nabla \mathcal{L}_\mu(\theta)/L \right)_i \right\}$  for any  $i \in \tilde{A}$ , and  $v_i = 0$  for any  $i \notin \tilde{A}$ . Then we have

$$\begin{aligned} v^\top \left( \nabla \mathcal{L}_\mu(\theta^*) - \nabla \mathcal{L}_\mu(\theta) \right) &= \sum_{i \in \tilde{A}} v_i \left( \nabla \mathcal{L}_\mu(\theta^*)/L - \nabla \mathcal{L}_\mu(\theta)/L \right)_i \\ &= \sum_{i \in \tilde{A}} \left| \left( \nabla \mathcal{L}_\mu(\theta^*) - \nabla \mathcal{L}_\mu(\theta) \right)_i \right| \geq \lambda |\tilde{A}|/2. \end{aligned} \quad (\text{N.26})$$

On the other hand, we have

$$\begin{aligned} v^\top \left( \nabla \mathcal{L}_\mu(\theta^*) - \nabla \mathcal{L}_\mu(\theta) \right) &\leq \|v\|_2 \|\nabla \mathcal{L}_\mu(\theta^*) - \nabla \mathcal{L}_\mu(\theta)\|_2 \stackrel{(i)}{\leq} \sqrt{|\tilde{A}|} \cdot \|\nabla \mathcal{L}_\mu(\theta^*) - \nabla \mathcal{L}_\mu(\theta)\|_2 \\ &\stackrel{(ii)}{\leq} \rho_{s^*+2\tilde{s}}^+ \sqrt{|\tilde{A}|} \cdot \|\theta - \theta^*\|_2, \end{aligned} \quad (\text{N.27})$$

where (i) is from  $\|v\|_2 \leq \sqrt{|\tilde{A}|} \max\{i : |A_i|\} \leq \sqrt{|\tilde{A}|}$ , and (ii) is from (A.1) and (A.2).

Combining (N.26) and (N.27), we have

$$\lambda |\tilde{A}| \leq 2\rho_{s^*+2\tilde{s}}^+ \sqrt{|\tilde{A}|} \cdot \|\theta - \theta^*\|_2 \stackrel{(i)}{\leq} 8\lambda \kappa_{s^*+2\tilde{s}} \sqrt{3s^* |\tilde{A}|}$$

where (i) is from (A.11) in Lemma A.3 and definition of  $\kappa_{s^*+2\tilde{s}} = \rho_{s^*+2\tilde{s}}^+ / \rho_{s^*+2\tilde{s}}^-$ . Considering  $A_3 \subseteq \tilde{A}$ , this implies

$$|A_3| \leq |\tilde{A}| \leq 196 \kappa_{s^*+2\tilde{s}}^2 s^*. \quad (\text{N.28})$$

Now combining Even  $A_1$ ,  $A_2$ ,  $A_3$  and  $L \leq 2\rho_{s^*+2\tilde{s}}^+$  in assumption, we close the proof as

$$\begin{aligned} \|\left( \mathcal{T}_{L,\mu,\lambda}(\theta) \right)_{\tilde{s}^*}\|_0 &\leq |A_1| + |A_2| + |A_3| \leq \frac{72Ls^*}{\rho_{s^*+2\tilde{s}}^-} + 196\kappa_{s^*+2\tilde{s}}^2 s^* \leq (144\kappa_{s^*+2\tilde{s}} + 196\kappa_{s^*+2\tilde{s}}^2) s^* \\ &\leq \tilde{s}. \end{aligned}$$

#### N.4 Proof of Lemma A.5

Let  $g = \text{argmin}_{g \in \partial \|\theta\|_1} \mathcal{L}_\mu + \lambda \|\theta\|_1$ , then  $\omega_{\mu,\lambda} = \|\nabla \mathcal{L}_\mu + \lambda g\|_\infty$ . By the optimality of  $\bar{\theta}$  and convexity of  $\mathcal{F}_{\mu,\tilde{\lambda}}$ , we have

$$\begin{aligned} \mathcal{F}_{\mu,\tilde{\lambda}}(\theta) - \mathcal{F}_{\mu,\tilde{\lambda}}(\bar{\theta}) &\leq \left( \nabla \mathcal{L}_\mu + \tilde{\lambda} g \right)^\top (\theta - \bar{\theta}) \leq \|\nabla \mathcal{L}_\mu + \lambda g\|_\infty \|\theta - \bar{\theta}\|_1 \\ &\leq \left( \omega_{\mu,\lambda}(\theta) + \lambda - \tilde{\lambda} \right) \|\theta - \bar{\theta}\|_1. \end{aligned} \quad (\text{N.29})$$

In addition, we have

$$\begin{aligned} \|\theta - \bar{\theta}\|_1 &\leq \|\theta - \theta^*\|_1 + \|\bar{\theta} - \theta^*\|_1 \stackrel{(i)}{\leq} 6 \left( \|(\theta - \theta^*)_{\mathcal{S}^*}\|_1 + \|(\bar{\theta} - \theta^*)_{\mathcal{S}^*}\|_1 \right) \\ &\leq 6\sqrt{s^*} \left( \|(\theta - \theta^*)_{\mathcal{S}^*}\|_2 + \|(\bar{\theta} - \theta^*)_{\mathcal{S}^*}\|_2 \right) \stackrel{(ii)}{\leq} \frac{12(\lambda + \tilde{\lambda})s^*}{\rho_{s^*+2\tilde{s}}^-}. \end{aligned} \quad (\text{N.30})$$

where (i) and (ii) are from (A.7) and (A.8) in Lemma A.2 respectively. Combining (N.29) and (N.30), we have the desired result.

## N.5 Proof of Lemma A.6

Our analysis has two steps. In the first step, we show that  $\{\theta^{(t)}\}_{t=0}^{\infty}$  converges to the unique limit point  $\bar{\theta}$ . In the second step, we show that the proximal-gradient method has linear convergence rate.

**Step 1.** Note that  $\theta^{(t+1)} = \mathcal{T}_{L^{(t+1)}, \mu, \lambda}(\theta^{(t)})$ . Since  $\mathcal{F}_{\mu, \lambda}(\theta)$  is convex in  $\theta$  (but not strongly convex), the sub-level set  $\{\theta : \mathcal{F}_{\mu, \lambda}(\theta) \leq \mathcal{F}_{\mu, \lambda}(\theta^{(0)})\}$  is bounded. By the monotone decrease of  $\mathcal{F}_{\mu, \lambda}(\theta^{(t)})$  from (A.16) in Lemma A.8,  $\{\theta^{(t)}\}_{t=0}^{\infty}$  is also bounded. By BolzanoWeierstrass theorem, it has a convergent subsequence and we will show that  $\bar{\theta}$  is the unique accumulation point.

Since  $\mathcal{F}_{\mu, \lambda}(\theta)$  is bounded below,

$$\lim_{k \rightarrow \infty} \|\theta^{(t+1)} - \theta^{(t)}\|_2 \leq \frac{2}{L_{\mu}^{(t)}} \cdot \lim_{k \rightarrow \infty} [\mathcal{F}_{\mu, \lambda}(\theta^{(t+1)}) - \mathcal{F}_{\mu, \lambda}(\theta^{(t)})] = 0.$$

By Lemma A.9, we have

$$\lim_{k \rightarrow \infty} \omega_{\mu, \lambda}(\theta^{(t)}) = 0,$$

This implies  $\lim_{k \rightarrow \infty} \theta^{(t)}$  satisfies the KKT condition, hence is an optimal solution.

Let  $\bar{\theta}$  be an accumulation point. Since  $\bar{\theta} = \arg\min_{\theta} \mathcal{F}_{\mu, \lambda}(\theta)$ , then there exists some  $g \in \partial \|\bar{\theta}\|_1$  such that

$$\nabla \mathcal{F}_{\mu, \lambda}(\bar{\theta}) = \mathcal{L}_{\mu, \lambda}(\bar{\theta}) + \lambda g = 0. \quad (\text{N.31})$$

By Lemma A.4, every proximal update is sparse, hence  $\|\bar{\theta}_{\bar{S}^*}\|_0 \leq \bar{s}$ . By RSC property in (3.1), if  $\|\theta_{\bar{S}^*}\|_0 \leq \bar{s}$ , i.e.,  $\|(\theta - \bar{\theta})_{\bar{S}^*}\|_0 \leq \bar{s}$ , then we have

$$\mathcal{L}_{\mu}(\theta) - \mathcal{L}_{\mu}(\bar{\theta}) \geq (\theta - \bar{\theta})^{\top} \nabla \mathcal{L}_{\mu}(\bar{\theta}) + \frac{\rho_{\bar{s}^* + 2\bar{s}}}{2} \|\theta - \bar{\theta}\|_2^2, \quad (\text{N.32})$$

From the convexity of  $\|\theta\|_1$  and  $g \in \partial \|\bar{\theta}\|_1$ , we have

$$\|\theta\|_1 - \|\bar{\theta}\|_1 \geq (\theta - \bar{\theta})^{\top} g. \quad (\text{N.33})$$

Combining (N.32) and (N.33), we have for any  $\|\theta_{\bar{S}^*}\|_0 \leq \bar{s}$ ,

$$\begin{aligned} \mathcal{F}_{\mu, \lambda}(\theta) - \mathcal{F}_{\mu, \lambda}(\bar{\theta}) &= \mathcal{L}_{\mu}(\theta) + \lambda \|\theta\|_1 - (\mathcal{L}_{\mu}(\bar{\theta}) + \lambda \|\bar{\theta}\|_1) \\ &\geq (\theta - \bar{\theta})^{\top} (\mathcal{L}_{\mu}(\bar{\theta}) + \lambda g) + \frac{\rho_{\bar{s}^* + 2\bar{s}}}{2} \|\theta - \bar{\theta}\|_2^2 \\ &\stackrel{(i)}{=} \frac{\rho_{\bar{s}^* + 2\bar{s}}}{2} \|\theta - \bar{\theta}\|_2^2 \geq 0, \end{aligned} \quad (\text{N.34})$$

where (i) is from (N.31). Therefore,  $\bar{\theta}$  is the unique accumulation point, i.e.  $\lim_{k \rightarrow \infty} \theta^{(t)} = \bar{\theta}$ .

**Step 2.** The objective  $\mathcal{F}_{\mu,\lambda}(\theta^{(t+1)})$  satisfies

$$\begin{aligned} \mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) &\stackrel{(i)}{\leq} \mathcal{Q}_{\mu,\lambda}(\theta^{(t+1)}, \theta^{(t)}) \\ &\stackrel{(ii)}{=} \min_{\theta} \mathcal{L}_{\mu}(\theta^{(t)}) + \nabla \mathcal{L}_{\mu}(\theta^{(t)})^{\top} (\theta - \theta^{(t)}) + \frac{\widetilde{L}_{\lambda}^{(t)}}{2} \|\theta - \theta^{(t)}\|_2^2 + \lambda \|\theta\|_1. \end{aligned} \quad (\text{N.35})$$

where (i) is from (A.16) in Lemma A.8, (ii) is from the definition of  $\mathcal{O}_{\mu,\lambda}$  in (2.3). To further bound R.H.S. of (N.35), we consider the line segment

$$S(\bar{\theta}, \theta^{(t)}) = \{\theta : \theta = \alpha \bar{\theta} + (1 - \alpha) \theta^{(t)}, \alpha \in [0, 1]\}.$$

Then we restrict the minimization over the line segment  $S(\bar{\theta}, \theta^{(t)})$ ,

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) \leq \min_{\theta \in S(\bar{\theta}, \theta^{(t)})} \mathcal{L}_{\mu}(\theta^{(t)}) + \nabla \mathcal{L}_{\mu}(\theta^{(t)})^{\top} (\theta - \theta^{(t)}) + \frac{\widetilde{L}_{\lambda}^{(t)}}{2} \|\theta - \theta^{(t)}\|_2^2 + \lambda \|\theta\|_1. \quad (\text{N.36})$$

Since  $\|\bar{\theta}_{\widetilde{S}^*}\|_0 \leq \widetilde{s}$  and  $\|\theta_{\widetilde{S}^*}^{(t)}\|_0 \leq \widetilde{s}$ , then for any  $\theta \in S(\bar{\theta}, \theta^{(t)})$ , we have  $\|\theta_{\widetilde{S}^*}\|_0 \leq \widetilde{s}$  and  $\|(\theta - \theta^{(t)})_{\widetilde{S}^*}\|_0 \leq 2\widetilde{s}$ . By RSC property, we have

$$\begin{aligned} \mathcal{L}_{\mu}(\theta) &\geq \mathcal{L}_{\mu}(\theta^{(t)}) + \nabla \mathcal{L}_{\mu}(\theta^{(t)})^{\top} (\theta - \theta^{(t)}) + \frac{\rho_{\widetilde{S}^*+2\widetilde{s}}^-}{2} \|\theta - \theta^{(t)}\|_2^2 \\ &\geq \mathcal{L}_{\mu}(\theta^{(t)}) + \nabla \mathcal{L}_{\mu}(\theta^{(t)})^{\top} (\theta - \theta^{(t)}). \end{aligned} \quad (\text{N.37})$$

Combining (N.36) and (N.37), we have

$$\begin{aligned} \mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) &\leq \min_{\theta \in S(\bar{\theta}, \theta^{(t)})} \mathcal{L}_{\mu}(\theta) + \frac{\widetilde{L}_{\lambda}^{(t)}}{2} \|\theta - \theta^{(t)}\|_2^2 + \lambda \|\theta\|_1 \\ &= \min_{\theta \in S(\bar{\theta}, \theta^{(t)})} \mathcal{F}_{\mu,\lambda}(\theta) + \frac{\widetilde{L}_{\lambda}^{(t)}}{2} \|\theta - \theta^{(t)}\|_2^2 \\ &= \min_{\alpha \in [0, 1]} \mathcal{F}_{\mu,\lambda}(\alpha \bar{\theta} + (1 - \alpha) \theta^{(t)}) + \frac{\alpha^2 \widetilde{L}_{\lambda}^{(t)}}{2} \|\bar{\theta} - \theta^{(t)}\|_2^2 \\ &\stackrel{(i)}{\leq} \min_{\alpha \in [0, 1]} \alpha \mathcal{F}_{\mu,\lambda}(\bar{\theta}) + (1 - \alpha) \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) + \frac{\alpha^2 \widetilde{L}_{\lambda}^{(t)}}{2} \|\bar{\theta} - \theta^{(t)}\|_2^2 \\ &\stackrel{(ii)}{\leq} \min_{\alpha \in [0, 1]} \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \alpha \left( \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \right) + \frac{\alpha^2 \widetilde{L}_{\lambda}^{(t)}}{\rho_{\widetilde{S}^*+2\widetilde{s}}^-} \left( \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \right) \\ &= \min_{\alpha \in [0, 1]} \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \alpha \left( 1 - \frac{\alpha \widetilde{L}_{\lambda}^{(t)}}{\rho_{\widetilde{S}^*+2\widetilde{s}}^-} \right) \left( \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \right), \end{aligned} \quad (\text{N.38})$$

where (i) is from the convexity of  $\mathcal{F}_{\mu,\lambda}$  and (ii) is from (N.34).



Minimize the R.H.S. of (N.38) w.r.t.  $\alpha$ , the optimal value  $\alpha = \frac{\rho_{s^*+2\bar{s}}^-}{2\bar{L}_\lambda^{(t)}}$  results in

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) \leq \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \frac{\rho_{s^*+2\bar{s}}^-}{4\bar{L}_\lambda^{(t)}} (\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})). \quad (\text{N.39})$$

Subtracting both sides of (N.39) by  $\mathcal{F}_{\mu,\lambda}(\bar{\theta})$ , we have

$$\begin{aligned} \mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) &\leq \left(1 - \frac{\rho_{s^*+2\bar{s}}^-}{4\bar{L}_\lambda^{(t)}}\right) (\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})) \\ &\stackrel{(i)}{\leq} \left(1 - \frac{\rho_{s^*+2\bar{s}}^-}{8\rho_{s^*+2\bar{s}}^+}\right) (\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})), \end{aligned} \quad (\text{N.40})$$

where (i) is from Remark A.1. Apply (N.40) recursively, we have the desired result.

## N.6 Proof of Lemma A.7

We first show an upper bound of  $L_\mu$ . Recall from the analysis in Appendix C of Lemma 3.6, there exists some  $\alpha \in [0, 1]$  such that

$$\nabla^2 \mathcal{L}_\mu = \begin{cases} \frac{X^\top X}{\sqrt{n}\mu}, & \text{if } \|xi\|_2 < \mu \\ \frac{1}{\sqrt{n}\|xi\|_2} X^\top \left(I - \frac{xi xi^\top}{\|xi\|_2^2}\right) X, & \text{o.w.} \end{cases}$$

where  $xi = y - X(w + \alpha\Delta)$ . We discuss two cases depending on  $\|xi\|_2 < \mu$  and  $\|xi\|_2 \geq \mu$ .

**Case 1.** For  $\|xi\|_2 < \mu$ , we have from the definition of  $L_\mu$  in Remark A.1 that

$$L_\mu \leq \|\nabla^2 \mathcal{L}_\mu\|_2 \leq \frac{\|X\|_2^2}{\sqrt{n}\mu}.$$

**Case 2.** For  $\|xi\|_2 \geq \mu$ , we have

$$L_\mu \leq \|\nabla^2 \mathcal{L}_\mu\|_2 = \frac{1}{\sqrt{n}\|xi\|_2} \left\| X^\top \left(I - \frac{xi xi^\top}{\|xi\|_2^2}\right) X \right\|_2 \leq \frac{\|X\|_2^2}{\sqrt{n}\|xi\|_2} = \frac{\|X\|_2^2}{\sqrt{n}\|xi\|_2} \leq \frac{\|X\|_2^2}{\sqrt{n}\mu}.$$

Combining the two cases, we have

$$L_\mu \leq \frac{\|X\|_2^2}{\sqrt{n}\mu}. \quad (\text{N.41})$$

Applying the analogous argument in Step 1 of the proof of Lemma A.6, we have that  $\{\theta^{(t)}\}_{t=0}^\infty$  converges to the unique limit point  $\bar{\theta}$ . By the monotonicity of  $\mathcal{F}_{\mu,\lambda}(\theta^{(t)})$  from (A.16) in Lemma A.8

and convexity of  $\mathcal{F}_{\mu,\lambda}(\theta)$ , we have  $\|\theta^{(t)} - \bar{\theta}\|_2 \leq R$  for all  $t = 1, 2, \dots$ . Then we have

$$\begin{aligned}
\mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) &\stackrel{(i)}{\leq} \mathcal{Q}_{\mu,\lambda}(\theta^{(t+1)}, \theta^{(t)}) \stackrel{(ii)}{\leq} \min_{\theta} \mathcal{F}_{\mu,\lambda}(\theta) + \frac{\widetilde{L}^{(t)}}{2} \|\theta - \theta^{(t)}\|_2^2 \\
&\leq \min_{\theta = \alpha \bar{\theta} + (1-\alpha)\theta^{(t)}, \alpha \in [0,1]} \mathcal{F}_{\mu,\lambda}(\theta) + \frac{\widetilde{L}^{(t)}}{2} \|\theta - \theta^{(t)}\|_2^2 \\
&= \min_{\alpha \in [0,1]} \mathcal{F}_{\mu,\lambda}(\alpha \bar{\theta} + (1-\alpha)\theta^{(t)}) + \frac{\widetilde{L}^{(t)}\alpha^2}{2} \|\theta^{(t)} - \bar{\theta}\|_2^2 \\
&\stackrel{(iii)}{\leq} \min_{\alpha \in [0,1]} \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \alpha (\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})) + \frac{2\|X\|_2^2 R^2 \alpha^2}{2\sqrt{n}\mu}, \tag{N.42}
\end{aligned}$$

where (i) and (ii) are from (A.16) and (A.15) in Lemma A.8 respectively, (iii) is from the convexity of  $\mathcal{F}_{\mu,\lambda}(\theta)$ ,  $\|\theta^{(t)} - \bar{\theta}\|_2 \leq R$  for all  $t = 1, 2, \dots$  and  $\widetilde{L}^{(t)} \leq 2L_\mu \leq 2\|X\|_2^2/(\sqrt{n}\mu)$  in Remark A.1 and (N.41). We discuss in two cases to provide an upper bound of R.H.S. (N.42).

**Case 1:** Suppose  $\mathcal{F}_{\mu,\lambda}(\theta^{(0)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \leq 2\|X\|_2^2 R^2/(\sqrt{n}\mu)$ . Minimizing the R.H.S. of (N.42) w.r.t.  $\alpha$ , then the optimal value is

$$\alpha = \frac{\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})}{2\|X\|_2^2 R^2/(\sqrt{n}\mu)} \leq \frac{\mathcal{F}_{\mu,\lambda}(\theta^{(0)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})}{2\|X\|_2^2 R^2/(\sqrt{n}\mu)} \leq 1.$$

Then we have

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) \leq \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \frac{(\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}))^2}{4\|X\|_2^2 R^2/(\sqrt{n}\mu)}$$

Equivalently, we have

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t+1)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \leq \mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) - \frac{(\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}))^2}{4\|X\|_2^2 R^2/(\sqrt{n}\mu)}$$

Denote  $f_k = \mathcal{F}_{\mu,\lambda}(\theta^{(k)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta})$ . Then we have

$$\frac{1}{f_{k+1}} \leq \frac{1}{f_k} - \frac{1}{4f_k^2\|X\|_2^2 R^2/(\sqrt{n}\mu)},$$

which results in

$$f_{k+1} \geq f_k + \frac{f_{k+1}}{4f_k\|X\|_2^2 R^2/(\sqrt{n}\mu)} \stackrel{(i)}{\geq} f_k + \frac{1}{4\|X\|_2^2 R^2/(\sqrt{n}\mu)}, \tag{N.43}$$

where (i) is from the monotonicity of  $\mathcal{F}_{\mu,\lambda}(\theta^{(t)})$  (A.16) in Lemma A.8. Applying (N.43) recursively, we have

$$f_k \geq f_0 + \frac{k}{4\|X\|_2^2 R^2/(\sqrt{n}\mu)} \stackrel{(i)}{\geq} \frac{t+2}{4\|X\|_2^2 R^2/(\sqrt{n}\mu)},$$

where (i) is from  $\mathcal{F}_{\mu,\lambda}(\theta^{(0)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) < 2\|X\|_2^2 R^2 / (\sqrt{n}\mu)$ . Then we have the desired result (A.14).

**Case 2:** Suppose  $\mathcal{F}_{\mu,\lambda}(\theta^{(0)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) > 2\|X\|_2^2 R^2 / (\sqrt{n}\mu)$ . Minimize the R.H.S. of (N.42) w.r.t.  $\alpha$ , then the optimal value is  $\alpha = 1$  and

$$\mathcal{F}_{\mu,\lambda}(\theta^{(1)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) \leq \frac{2\|X\|_2^2 R^2}{2\sqrt{n}\mu}.$$

We claim that for all  $t = 1, 2, \dots$ ,

$$\mathcal{F}_{\mu,\lambda}(\theta^{(t)}) - \mathcal{F}_{\mu,\lambda}(\bar{\theta}) = c_t 2\|X\|_2^2 R^2 / (\sqrt{n}\mu),$$

where  $\frac{1}{t+2} \leq c_t \leq \frac{2}{t+2}$ . We prove the claim by induction.

This obviously holds when  $t = 1$ . Assume  $\frac{1}{t+2} \leq c_t \leq \frac{2}{t+2}$  holds when  $t = T$ . For  $t = T + 1$ , minimize the R.H.S. of (N.42) w.r.t.  $\alpha$ , then the optimal value is  $\alpha = c_T \leq 1$  and convergence rate for  $T + 1$ -th iteration is  $c_{T+1} = c_T - c_T^2/2$ . Since  $c_{T+1}$  is a increasing function of  $c_T$  in  $c_T \in [0, 1/2]$  and  $c_t$  is monotone decreasing, i.e.,  $c_t \leq c_1$  for all  $k > 1$ , then we verify the claim since

$$c_{T+1} \leq \frac{2}{T+2} - \frac{1}{2} \left( \frac{2}{T+2} \right)^2 \leq \frac{2}{T+3} \quad \text{and} \quad c_{T+1} \geq \frac{1}{T+2} - \frac{1}{2} \left( \frac{1}{T+2} \right)^2 \leq \frac{1}{T+3}.$$

Combining the two cases, we have the desired result (A.14).

## N.7 Proof of Lemma K.4

Recall that the model is

$$z_i = Z_{*\setminus i} \theta_i^* + \widehat{\Gamma}_{ii}^{-1/2} \epsilon_i,$$

where  $z_i = \widehat{\Gamma}_{ii}^{-1/2} x_i$ ,  $Z_{*\setminus i} = X_{*\setminus i} \widehat{\Gamma}_{i\setminus i}^{-1/2}$  and  $\epsilon_i \sim \mathcal{N}(0, \sigma_i^2 I_n)$ . Then we have

$$\nabla \mathcal{L}_{\mu,i}(\theta_i^*) = \frac{Z_{*\setminus i}^\top (Z_{*\setminus i} \theta_i^* - z_i)}{\max\{\sqrt{n}\mu, \sqrt{n}\|z_i - Z_{*\setminus i} \theta_i^*\|_2\}} = - \frac{Z_{*\setminus i}^\top \epsilon_i}{\max\{\sqrt{n}\mu \widehat{\Gamma}_{ii}^{1/2}, \sqrt{n}\|\epsilon_i\|_2\}}.$$

Since  $\frac{\|\epsilon_i\|_2}{n\sigma_i^2} \sim \chi_n^2$ , we have from Johnstone (2001) that for any  $\delta \in [0, 1/2)$ ,

$$\mathbb{P} \left[ \max_{i \in \{1, \dots, d\}} \frac{\|\epsilon_i\|_2^2}{n\sigma_i^2} \leq 1 - \delta \right] \leq d \exp \left( -\frac{n\delta^2}{4} \right). \quad (\text{N.44})$$

In addition,  $Z_{*\setminus i}^\top \epsilon_i \sim \mathcal{N}(0, n\sigma_i^2)$ . Then we have from Liu and Wang (2012) that for any  $\delta \in [0, 1/2)$  and  $c > 2$ ,

$$\mathbb{P} \left[ \max_{i \in \{1, \dots, d\}} \|Z_{*\setminus i}^\top \epsilon_i\|_\infty > \sigma_i \sqrt{2cn \log d(1-\delta)} \right] \leq \frac{d^{2-c(1-\delta)}}{\sqrt{\pi a \log d(1-\delta)}}. \quad (\text{N.45})$$

Combining (N.44) and (N.45), we have with probability at least  $1 - d \exp \left( -\frac{n\delta^2}{4} \right) - \frac{d^{2-c(1-\delta)}}{\sqrt{\pi a \log d(1-\delta)}}$ ,

$$\max_{i \in \{1, \dots, d\}} \frac{\|Z_{*\setminus i}^\top \epsilon_i\|_\infty}{\max\{\widehat{\mu} \widehat{\Gamma}_{ii}^{1/2}, \sqrt{n}\|\epsilon_i\|_2\}} \leq \frac{\sqrt{2c \log d(1-\delta)}/n}{\max\{\min_{i \in \{1, \dots, d\}} \widehat{\mu} \widehat{\Gamma}_{ii}^{1/2} / (\sqrt{n}\sigma_i), \sqrt{1-\delta}\}}.$$

Take  $\delta = 2/5$  and  $c = 7/3$ , then we have the desired result.